Entanglement with an accelerated observer

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1 Basic situation

We work in 1+1 dimensions: one space dimension and time.
Alice and Bob stand in the laboratory, and share an EPR state:

$$\psi_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right). \tag{1}$$

What does this mean: $|0\rangle$ means there is no probability to observe a particle. $|1\rangle$ means there is a probability to observe one particle.
We work with momentum, so that $|1\rangle$ means probability to observe one particle with momentum k somewhere in all of space. (we could write $|k\rangle$ but we will assume only one possible momentum, for simplicity.)

1.1 The formalism

We work in Fock space. This means that we have creation operators so that

$$a^\dagger |0\rangle = |1\rangle , \; a |0\rangle = 0. \tag{2}$$

To create two particles (with the same momentum) we use the operator twice:

$$a^\dagger a^\dagger |0\rangle = a^\dagger |1\rangle = 2 \tag{3}$$

and to creat n particles we do it n times:

$$\left(a^\dagger\right)^n |0\rangle \sim |n\rangle . \tag{4}$$

Proportional (not equal) because

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \rightarrow \left(a^\dagger\right)^n |0\rangle = \sqrt{n!} |n\rangle . \tag{5}$$

That means we have n particles with the given momentum, which could be found somewhere in all of space.
In Alice and Bob's EPR state, it means they share an amplitude that either they will find no particles, or each will find one particle somewhere in space.You could say they each hold a detector which can register whether or not there's a particle somewhere in space with the given momentum. Either both detectors will click, or neither of them will click.

1.2 Entanglement

While Alice and Bob stand in the lab, we check entanglement. Tracing out Alice, the reduced density matrix is

$$\begin{aligned} \rho_B &= \langle 0_A |\psi_{AB}\rangle \langle\psi_{AB}| 0_A\rangle + \langle 1_A |\psi_{AB}\rangle \langle\psi_{AB}| 1_A\rangle \\ &= \frac{1}{2} \left(|0_B\rangle \langle 0_B| + |1_B\rangle \langle 1_B| \right) \end{aligned} \tag{6}$$

that is, $\frac{1}{2}$ of the 2 dimensional identity matrix. Using Von Neuman entropy,

$$S_{VN} = -Tr\left(\rho_B log \rho_B\right) = -\sum_{i=1}^{dim} \lambda_i log_2 \lambda_i \tag{7}$$

where λ_i are eigenvalues of the reduced density matrix, we get $S_{VN} = -2(\frac{1}{2}log_2\left[\frac{1}{2}\right]) = log_2 2 = 1$ and see that the state is maximally entangled.
The question: if Bob gets into a car and drives off with constant acceleration - will he still find that the state is maximally entangled? That either both detectors will click, or neither will, with equal amplitude?

2 Bob accelerates

Bob gets into a car and speed up with constant acceleration a . This means he is now in an accelerated coordinate system ("Rindler space").

2.1 Coordinates:

We first find his trajectory. This is done in the "co-moving frame" where Bob is instantaneously at rest, so that his path can be expressed in the lab coordinates.
Bob moves at uniform acceleration of magnitude α in the x direction. His path is

$$\begin{aligned} x^{\mu}(\tau) &= \left(t\left(\tau\right), x\left(\tau\right)\right) \\ t\left(\tau\right) &= \frac{1}{\alpha} sinh(\alpha\tau),\ x(\tau) = \frac{1}{\alpha} coshc(\alpha\tau) \end{aligned} \tag{8}$$

To see why, differentiate x^{μ} twice, and you will get the constant acceleration.

$$\begin{aligned} u^{\mu}(\tau) &= \frac{dx^{\mu}}{d\tau} = (cosh(\alpha\tau), sinh(\alpha\tau)) \\ a^{\mu}(\tau) &= \frac{du^{\mu}}{d\tau} = \left(\alpha sinh\left(\alpha\tau\right), \alpha cosh\left(\alpha\tau\right)\right) \\ \sqrt{a^{\mu}a_{\mu}} &= \sqrt{g_{\mu\mu}a^{\mu}a^{\mu}} = \sqrt{\alpha^2\left(-sinh^2\left(\alpha\tau\right) + cosh^2\left(\alpha\tau\right)\right)} = \alpha. \end{aligned} \tag{9}$$

You also see that $x^2 - t^2 = \frac{1}{a^2}$, a hyperbola. Figure 1 shows Bob's world line in Minkowsky space, assuming he's going in the positive x direction. If he were going in the $-x$ direction the graph would be a mirror image on the left of the t axis.

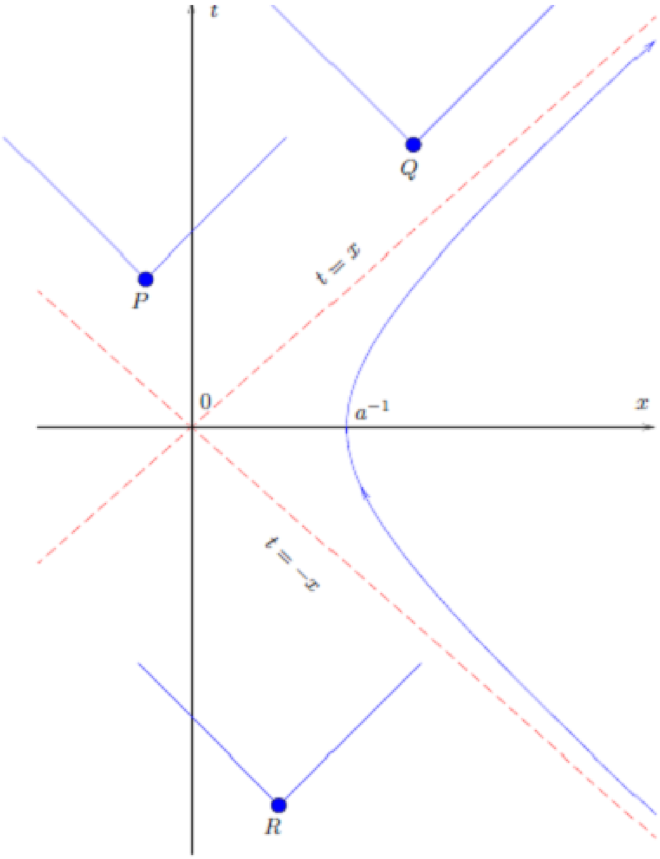

Figure 1: The worldline of a uniformly accelerated observer (proper acceleration $a \equiv |\mathbf{a}|$) in the Minkowski spacetime. The dashed lines show the light-cone. The observer cannot receive any signals from the events P, Q and cannot send signals to R.

Bob doesn't measure things in the lab frame but in his own "proper" frame. We call Bob's coordinates η (his time) and ξ (his space coordinate). The derivation is at the end, now we just write down the relation between t, x and η, ξ :

$$\begin{aligned} t &= \frac{1}{\alpha} e^{\alpha\xi} sinh\left(\alpha\eta\right) \\ x &= \frac{1}{\alpha} e^{\alpha\xi} cosh\left(\alpha\eta\right) \\ ds^2 &= e^{2\alpha\xi}\left(-d\eta^2 + d\xi^2\right) \end{aligned} \tag{10}$$

This is known as the Rindler coordinate system. Note that time and space coordinates mix: Minkowsky time contains Rindler space and time, so does Minkowsky space, and vice versa.
Also note that Rindler space only covers part of Minkowsky space because the speed of light is finite. So there is a part of space from whch no signal can reach Bob.

2.2 Bob's state

Since his space and time coordinates are both different, his "measuring ruler" for finding particles is different. Details are in Appendix A. For now I will just say what the difference is.
There are 2 important differences: First, the ground state in the lab now seems to Bob to have particles in it:

$$|0\rangle_{lab} \to |n\rangle_{Bob} . \tag{11}$$

Second, the “size” of the particles is different. This is because the time and space coordinates have changed, so the momentum has changed accordingly. Therefore we have a different creation operator:

$$b^{\dagger} \left|0\right\rangle_{Bob} \;=\; \left|1\right\rangle_{Bob} \tag{12}$$

and $b^{\dagger} \neq a^{\dagger}$.

So even without Alice, if Bob has his own state (for instance he holds a hydrogen atom), if it looked like a ground state to him in the lab, it will now look like an excited state. And if it looked like an excited state in the lab, it looks like a *different* excited state in the car (more excited, and having a different amount of energy between each level than he had in the lab.)

2.3 Translation of lab state into Bob's coordinates.

Now it becomes more complicated yet.

In the lab we had two possibilities, $\left|0\right\rangle_{lab}$ and $\left|1\right\rangle_{lab}$. They refer to the possibility to find particles anywhere in space. But Bob has to describe this (all space) in two parts: the part to which he has access, (the right side on the drawing) and the part to which he does not. So actually eq.11 should be written

$$\left|0\right\rangle_{lab} \rightarrow \left|\psi_I\right\rangle \left|\psi_{II}\right\rangle_{Bob}$$

where I, II refer to the left and right halves of Minkowsky space, and we have to separate the parts which Bob can or cannot know about. We expand the lab state in Bob's basis of $\left|n_I\right\rangle \left|n_{II}\right\rangle_{Bob}$. It's a vacuum in the lab, but Bob sees it as a superposition of excited states in his basis. (remember, an excited state in Bob's basis isn't the same as an excited state in the lab.)

It can be shown [2] that

$$\begin{aligned} \left|0\right\rangle_{lab} &= \frac{1}{cosh(r)} \sum_{n=0}^{\infty} tanh^{n}\left(r\right) \left|n_I\right\rangle \left|n_{II}\right\rangle_{Bob} \\ cosh\left(r\right) &= \frac{1}{\sqrt{1 - exp\left(-2\pi\Omega\right)}} \qquad (13) \\ \Omega &\equiv \frac{|k|}{a} \qquad (14) \end{aligned}$$

Cosh[r] rises monotonically from 0, while Tanh[r] has a maximum of 1. A plot of r as a function of the acceleration, for constant k, shows that r rises as a increases:

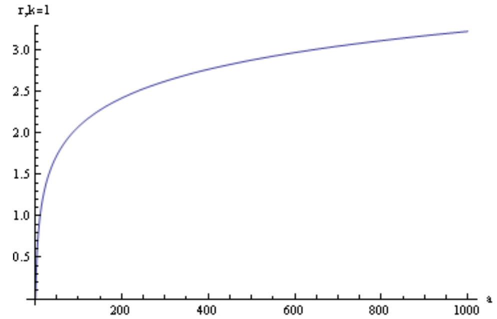

Figure 2:

For small acceleration, $cosh(r) \approx 1$, as acceleration increases $cosh(r)$ increases

We say Bob is in the right half, labeled I. Then Bob's excited state will be

$$\left|1\right\rangle_{lab} = \frac{1}{cosh(r)} \sum_{n=0}^{\infty} tanh^{n}\left(r\right) \sqrt{n+1} \left|n+1_I\right\rangle \left|n_{II}\right\rangle_{Bob}. \tag{15}$$

We put the excited state in Bob's part of space because that describes his amplitude to detect a photon. [1] But since we are translating the lab state into Bob's coordinates, we have to write$|n_{II}\rangle$ as well, because the lab state related to ALL of space, not just Bob's part of it.
In the lab the density matrix for Alice and Bob's state was

$$\begin{aligned} |\psi_{AB}\rangle\langle\psi_{AB}| &= \frac{1}{2}\left[|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\right]\left[\langle 0_A|\langle 0_B| + \langle 1_A|\langle 1_B|\right] \\ &= \frac{1}{2}\left[|0_A\rangle|0_B\rangle\langle 0_A|\langle 0_B| + |0_A\rangle|0_B\rangle\langle 1_A|\langle 1_B| + |1_A\rangle|1_B\rangle\langle 0_A|\langle 0_B| + |1_A\rangle|1_B\rangle\langle 1_A|\langle 1_B|\right]. \end{aligned}$$

We now rewrite this in terms of lab states for Alice and car states for Bob,

$$|m_A\rangle|n_B\rangle$$

Using eq.11 we might think

$$|0_A\rangle|0_B\rangle = |0_A\rangle\frac{1}{cosh(r)}\sum_{n=0}^{\infty} tanh^n(r)|n_I\rangle|n_{II}\rangle_{Bob}. \tag{16}$$

But this is a problem because Bob has no access to area II . So he can't write his state in terms of $|n_I\rangle|n_{II}\rangle$. He's in a mixed state where he lacks information about area II, and therefore has to trace out that area and write a density matrix. The resulting density matrix for Alice and Bob will have components such as

$$|0_A\rangle|n_{I,B}\rangle\langle 0_A|\langle n_{I,B}|$$

and so on. (without n_{II}). From now on we will write this as $|0,n\rangle\langle 0,n|$ for simplicity (without the subscripts), where the first part refers to Alice, in the lab, and the second to Bob, in the half of space which he has access to, after tracing out n_{II}:

$$\begin{aligned} \rho_{AB} &= \frac{1}{2cosh^2 r}\sum_{n=0}^{\infty}\left(tanh(r)\right)^{2n}\rho_n \\ \rho_n &= |0,n\rangle\langle 0,n| + \frac{\sqrt{n+1}}{cosh(r)}|0,n\rangle\langle 1,n+1| + \\ &\quad \frac{\sqrt{n+1}}{cosh(r)}|1,n+1\rangle\langle 0,n| + \frac{n+1}{cosh^2(r)}|1,n+1\rangle\langle 1,n+1|. \end{aligned} \tag{17}$$

3 Entanglement

Here we can't use Von Neumann entropy because Bob is in a mixed state even before tracing out Alice. For a mixed state in general

$$\rho_{AB} = \sum_{ijkl} W_{ijkl}|i\rangle|j\rangle\langle k|\langle l| \tag{18}$$

you don't have a Schmidt decomposition. Instead we use the partial transpose criterion for entanglement (valid for bipartite state). Given a composite state ρ_{AB}, if it is a product state

$$\rho_{AB} = \sum_{i,j} A_{ij}\rho_A^i \otimes \rho_B^j$$

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[1] Actually the reason is that Alice and Bob have the same time orientation. What Alice considers a particle, Bob also considers a particle, whereas if Bob were in the other part of space he'd see it as an antiparticle.

it will decompose into positive density matrices even if you transpose components of the subsystems. If it's entangled you can't decompose it. So if at least one eigenvalue of the partially transposed density matrix is negative, the density matrix is entangled. (Further details in the Appendix.)
After doing partial transpose, we look at the eigenvalues in the $n, n+1$ block of the transposed matrix (since n goes to infinity, we have to choose one block):

$$\lambda_{\pm}^{n} = \frac{tanh^{2n} r}{4cosh^2 r}\left[\left(\frac{n}{sinh^2 r} + tanh^2 r\right) \pm \sqrt{Z_n}\right] \tag{19}$$

$$Z_n = \left(\frac{n}{sinh^2 r} + tanh^2 r\right)^2 + \frac{4}{cosh^2 r}. \tag{20}$$

If the acceleration is finite ($r < \infty$) one eigenvalue is always negative. So the state is always entangled. The question is: is it maximally entangled as before?
To find out just how entangled it is we sum over all the negative eigenvalues and calculate the logarithmic negativity.
Logarithmic negativity is defined as follows, where ρ^{pt} is the partially transposed ρ_{AB} density matrix:

$$\begin{aligned} LN\left(\rho_{AB}\right) &= log_2 \left\|\rho^{pt}\right\| \\ \left\|\rho^{pt}\right\| &= \sqrt{\rho^{pt}\left(\rho^{pt}\right)^T} \end{aligned} \tag{21}$$

(trace norm of the density matrix). It relates to the negativity thus

$$\begin{aligned} N &= \sum_i \frac{|\lambda_i| - \lambda_i}{2} \\ LN &= log_2\left[2N+1\right]. \end{aligned}$$

For instance, for an EPR state in the lab, as Alice and Bob had at first, there is 1 negative eigenvalue in the partially transposed matrix, so the negativity is 1, and the logarithmic negativity is $log_2 3 \approx 1.585$. When Bob is accelerating this becomes a function of the acceleration parameter r:

$$\begin{aligned} LN(\rho_{AB}) &= log_2\left(\frac{1}{2cosh^2 r} + S\right) \\ S &= \sum_{n=0}^{\infty} \frac{tanh^{2n} r}{2cosh^2 r}\sqrt{\left(\frac{n}{sinh^2 r} + tanh^2 r\right)^2 + \frac{4}{cosh^2 r}}. \end{aligned} \tag{22}$$

For vanishing acceleration $r = 0$, $N = 1$ and entanglement is maximal like before. For finite acceleration entanglement is smaller. Figure 2 is a graph of the logarithmic negativity as a function of the acceleration parameter r:

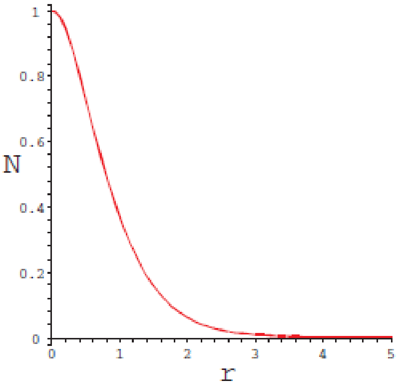

Figure 3:

. We see that the degree of entanglement is less as the acceleration grows. This means that entanglement is an observer dependent quantity. The physical situation of the particles hasn't changed, but Bob sees them as less entangled than Alice does: when we write them using Bob's basis the entanglement shrinks as a function of the acceleration.

A Unruh effect

A.1 Rindler coordinates

A.1.1 Comoving frame:

There are 3 coordinate frames involved: the lab frame where Alice stands, the co-moving frame which freezes Bob for an instant, and the proper frame where Bob lives. (Of course for each t there's a different comoving frame). We write the lab coordinates as functions of his proper time τ as follows. We know the following:

$$u^{\mu} \equiv \frac{dx^{\mu}}{d\tau},\ u_{\mu}u^{\mu} = 1,\ a_{\mu}a^{\mu} = |a|^2$$

We assume $u^0 > 0$ and $du^1/d\tau > 0$ (acceleration is in positive x direction) and so, dropping the y, z space dimensions we have

$$u^{\mu} = \left(u^0, u^1\right),\ a^{\mu} = \frac{du^{\mu}}{d\tau}$$

$$\left(u^0\right)^2 + \left(u^1\right)^2 = 1, \qquad \left(\frac{du}{d\tau}^0\right)^2 + \left(\frac{du}{d\tau}^1\right)^2 = \alpha^2.$$

$$u^0 = \sqrt{1 + (u^1)^2}, \qquad \frac{du^1}{d\tau} = \alpha\sqrt{1 + (u^1)^2} \tag{23}$$

Taking initial condition as $u^1(0) = 0$, integration gives

$$u^1 = sinh(a\tau)$$
$$u^0 = cosh(a\tau).$$

Integrate again, taking $\tau(0) = 0$:

$$\begin{aligned} x &= \frac{1}{\alpha} cosh(a\tau) - \frac{1}{\alpha} + x_0 \\ t &= \frac{1}{\alpha} sinh(a\tau). \end{aligned} \tag{24}$$

For convenience we choose initial conditions $x_0 = \frac{1}{\alpha}$ to obtain the results given in Sec.2.1.

A.1.2 Proper frame

Lab coordinates are x. The coordinates above were for the comoving frame, instantaneously coinciding with the accelerating system. We want the coordinates seen by an inertial observer in the lab (or floating in space as the ship passes). So we do a Lorentz transform from the comoving frame to the lab frame.
We call Bob's coordinates τ, ξ . Since he doesn't see himself as moving, he is always at $\xi = 0$. To work out the relationship of τ, ξ to the lab x, t we give him a stick to hold out, so one end of it is in his hand at $\xi = 0$ and the other end at some particular ξ. The stick moves with Bob, so the coordinates at the end of the stick are (τ, ξ) at some time τ (ignoring the y, z coordinates). The comoving frame is an inertial system moving at $u^\mu = dx^\mu / d\tau$. So we can make an inverse Lorentz transformation to relate the comoving system to the lab. Call the lab frame s, and the moving frame $\tilde{s}$. Then $s^\mu = \Lambda^\mu_\nu \tilde{s}^\nu$:

$$\begin{aligned} \begin{pmatrix} s^0 \\ s^1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} \tilde{s}^0 \\ \tilde{s}^1 \end{pmatrix} \\ &= \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix} \begin{pmatrix} \tilde{s}^0 \\ \tilde{s}^1 \end{pmatrix} = \begin{pmatrix} u^1 \xi \\ u^0 \xi \end{pmatrix} \end{aligned}$$

because $\tilde{s}^0 = 0$, $s^1 = \xi$. So the lab coordinates will be

$$\begin{aligned} t(\tau, \xi) &= x^0(\tau) + s^0_{lab} = \quad x^0(\tau) + \frac{dx^1(\tau)}{d\tau}\xi \\ x(\tau, \xi) &= x^1(\tau) + s^1_{lab} = \quad x^1(\tau) + \frac{dx^0(\tau)}{d\tau}\xi. \end{aligned} \tag{25}$$

Now we plug in $x^\mu(\tau)$ from eq.(24) and their derivatives, with the initial conditions as above $x^0(0) = 0$, $x^1(0) = \frac{1}{\alpha}$, and obtain

$$\begin{aligned} t(\tau, \xi) &= \left(\frac{1}{\alpha} + \xi\right) sinh(\alpha\tau) \\ x(\tau, \xi) &= \left(\frac{1}{\alpha} + \xi\right) cosh(\alpha\tau) \end{aligned} \tag{26}$$

and the inverse coordinates are

$$\begin{aligned} \tau(t, x) &= \frac{1}{2\alpha} ln \frac{x+t}{x-t} \\ \xi(t, x) &= -\frac{1}{\alpha} + \sqrt{x^2 + t^2}. \end{aligned} \tag{27}$$

Limits are $\xi > -1/\alpha$.

A.1.3 Causality

Note that a signal sent from the origin $(x = 0, t = 0)$ will reach coordinate $\xi = -\frac{1}{\alpha}$. If we assume Bob to be at $\xi = 0$, it will never reach him. For any fixed t there is thus an entire region inaccessible to Bob.

A.1.4 Redefinition of metric

The metric now is

$$ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 \, d\tau^2 - d\xi^2. \tag{28}$$

However - For ease of calculations we want a metric that is conformally flat (you can only do it in 2d). Define $\tilde{\xi}$ so

$$\begin{aligned} d\xi &= \left(\frac{1}{\alpha} + \xi\right) d\tilde{\xi} \\ \tilde{\xi} &= \frac{1}{\alpha} ln \left(\frac{1}{\alpha} + \xi\right). \end{aligned} \tag{29}$$

Since $\xi > -1/\alpha$, limits for $\tilde{\xi}$ are $\pm\infty$. Rename the time coordinate η to avoid confusion with t. The coordinates are now

$$\begin{aligned} t &= \frac{1}{\alpha} e^{\alpha\xi} sinh\left(\alpha\eta\right) \\ x &= \frac{1}{\alpha} e^{\alpha\xi} cosh\left(\alpha\eta\right) \end{aligned} \tag{30}$$

and the metric is

$$ds^2 = e^{2\alpha\xi}\left(d\eta^2 - d\xi^2\right) \tag{31}$$

which is conformal to Minkowsky space.

A.2 Operators

Rindler space operators are developed using quantum field theory. A field is written with both creation and destruction operators: $\psi(p) = ae^{-ip_\mu x^\mu} + cc$. Since now we have different x, t and thus different momenta and frequency for the Rindler system, the field operators are different from the Minkowsky operators, as follows

$$\begin{aligned} \psi_{mink}(p) &\sim ae^{ipx - iwt} + a^\dagger e^{-ipx + iwt}, a_{mink} \left|0\right\rangle_m = 0 \\ \psi_{Rind}(p) &\sim be^{i\tilde{p}\xi - i\tilde{w}\eta} + b^\dagger e^{-i\tilde{p}\xi + i\tilde{w}\eta}, b_{Rind} \left|0\right\rangle_R = 0. \end{aligned} \tag{32}$$

Since x and t *each* contain both ξ and η , therefore $iwt \neq i\tilde{w}\eta$. Finding the relation between the two frequencies is a complicated problem in quantum field theory. The main point is that since the field operators ψ are different, so are the creation and destruction operators, and so $a \neq b$.
To find the relation we expand the Rindler operator in terms of the Minkowsky ones:

$$b_{\tilde{w}} = \int_0^\infty dw \left[\alpha_{\tilde{w}\omega} a_\omega - \beta_{\tilde{w}\omega} a_\omega^\dagger\right] \tag{33}$$

or in the discrete case this would be

$$b_i = \sum_{j=0}^{\infty} \left[\alpha_{ij} a_j - \beta_{ij} a_j^\dagger\right]. \tag{34}$$

We can work out the relation between α and β . The normalization condition is

$$\int_0^\infty dw \left(\alpha_{\tilde{w}\omega} \alpha^*_{\tilde{w}'\omega} - \beta_{\tilde{w}\omega} \beta^*_{\tilde{w}'\omega}\right) = \delta\left(\tilde{w} - \tilde{w}'\right). \tag{35}$$

The procedure is to write down ψ in each set of coordinates and then plug in eq.33 and equate them. Then extract the coordinates of the ladder operator a from each side and equate them. This is tricky because the measure of the integral includes $\sqrt{g}$ from the Rindler metric, and it affects the Fourier transform as well. (This is how the acceleration enters into the final result). For details see, for instance, [4],[5]. You wind up with

$$|\alpha|^2 = e^{2\pi\tilde{w}/a}\,|\beta|^2\,. \tag{36}$$

(The a in the exponent refers to acceleration.) The number of particles a Rindler observer sees in a Minkowsky vacuum is

$$N = {}_M\langle O|\, b^\dagger b\,|0\rangle_M = \int d\omega|\beta|^2 = \frac{1}{e^{2\pi\tilde{w}/a}-1}\delta(0). \tag{37}$$

The delta function is the infinite volume of space, and diving by that we get the particle density:

$$n = \frac{N}{V} = \frac{1}{e^{2\pi\tilde{w}/a}-1} \tag{38}$$

This is just the Planck thermal distribution, with inverse temperature $\beta = 2\pi/a$ and so the Unruh temperature is

$$T = \frac{a}{2\pi}. \tag{39}$$

B Partial transpose criterion

Take the composite density matrix and transpose the components of one of the subsystems. If the resulting matrix has positive eigenvalues, the system is separable and was composed of non entangled systems. To understand why, write out the matrix elements:

$$\rho_{m\mu,n\nu} = \sum_r p_r(\rho^r_A)_{mn}(\rho^r_B)_{\mu\nu} \tag{40}$$

Latin indices refer to Alice, Greek indices to Bob. The partial transpose is given by transposing only one of the subsystems. The entries of the density matrix are then

$$\rho^{PT}_{m\mu,n\nu} = (\rho^T_A)_{m\mu,n\nu} = \rho_{n\mu,m\nu} \tag{41}$$

If the joint density matrix is separable, it's a tensor product of two density matrices each non negative with unit trace:

$$\rho^{PT}_{sep} = \sum_i p_i(|A_i\rangle\langle A_i|)^T \otimes |B_i\rangle\langle B_i| = \sum_i p_i(\rho^A_i)^T \otimes \rho^B_i$$

The transposed matrices $(\rho^A)^T = (\rho^A)^*$ are also non-negative with unit trace, and so none of the eigenvalues of ρ^{PT} is negative. If ρ^{PT} has negative eigenvalues, it means it was not derived from a separable product of two density matrices, and so they were entangled. If the eigenvalues are positive, it has been shown that the density matrix might be entangled but the entanglement can't be distilled and used; this is called bound entanglement.

References

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