Quantum Zeno Effect, Anti Zeno Effect and the Quantum recurrence theorem

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Quantum recurrence theorem:

The Quantum recurrence theorem [1] states that for every system with discrete energy spectrum, initial state $|\psi(0)\rangle$ and $\epsilon > 0$ there exits a time t_n for which

$$||\psi(0) - \psi(t_n)||^2 < \epsilon \tag{1}$$

Our goal is to find $t_n(\epsilon)$.

We denote the system's enrgy level $E_m = 2\pi\hbar\nu_m$ and a general wave function as $\psi = \sum c_m u_m e^{-ie_m t}$, where $\mathcal{H}u_m = E_m u_m$. Placing into (1) we, again, derive our goal: Find $t_n(\epsilon)$ for which:

$$4\sum |c_m|^2 \sin^2(\pi\nu_m t_n) < \epsilon \tag{2}$$

If we assume that $c_m = 1/N$ for every m [Side note 1]. Since we take $\epsilon << 1$ we have integers k_m for which we can simplify equation (2) using the aforemetioned assumption to:

$$\sum(\nu_m t_n - k_m) < N\frac{\epsilon}{4\pi^2} \tag{3}$$

This equation represents a sphere in k-space, such as the one demonstrated for the $N = 2$ case in Fig. 1 (in this case a circle):

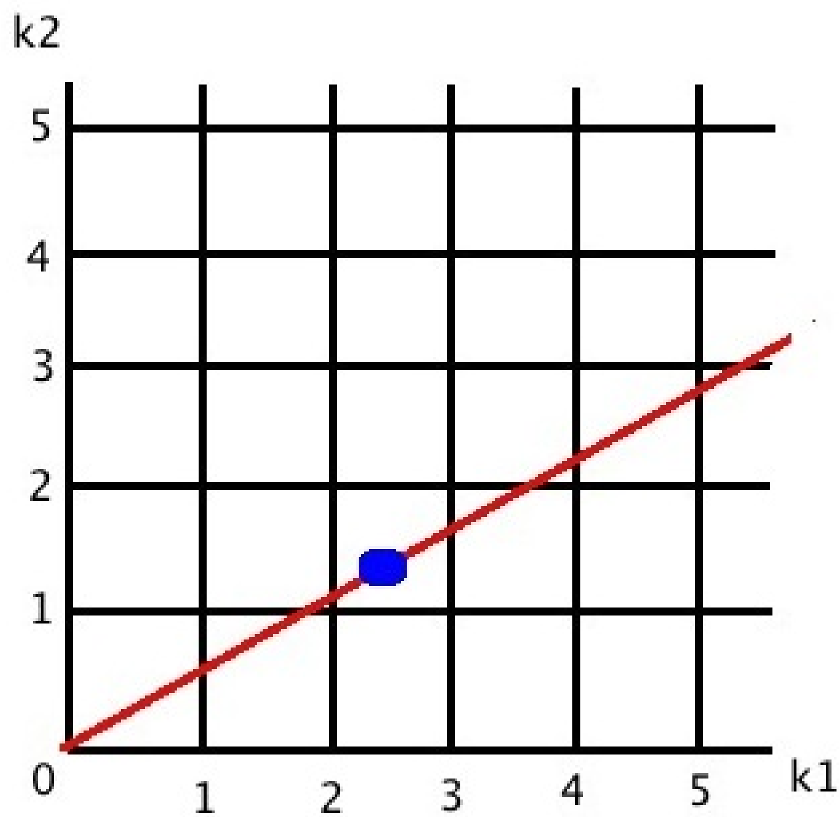

Fig. 1: Two dimentional representation of equation (3). k_1 and k_2 form a unit lattice. The red line signifies the trajectory that $\nu_m t$ takes and the blue circle represents the sphere with radius $N\epsilon/4\pi^2$.

With time, this sphere sweeps a cylinder of cross-section:

$$\sigma = \pi^{(N-1)/2} R^{N-1} / \Gamma[(N+1)/2] \tag{4}$$

Where $R = \sqrt{N\epsilon}/2\pi$, and

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \overset{z=1,2,\dots}{=} (z-1)! \tag{5}$$

For the 3D case, for example, we have $\sigma_{3D} = \pi R^2$ which is what we would expect to find.
The length of the cylinder is

$$(\sum \nu_m^2)^{1/2} t = N^{1/2} \nu_{rms} t \tag{6}$$

From here, we can expect to find a unit lattice until we sweep a unit cube so

$$t_n N^{1/2} \sigma \nu_{rms} = 1 \tag{7}$$

and

$$t_n \approx \frac{\Gamma[(N+1)/2]}{\nu_{rms} N^{1/2}} (\frac{4\pi^3}{N\epsilon})^{\frac{N-1}{2}} \tag{8}$$

*Side note 1 - if we hadn't taken $c_m = 1/N$ we would have gotten an ellipsoid instead of a sphere, yet our results still would have been valid, as explained in [1].
*Side note 2 - taking N to be finitie is justified by the fact that $\sum |c_m|^2 = 1$, thus we can find N for which this sum (truncated at N) is very close to 1 (taking appropriate ϵ).

Next, we take a look at the quantum Zeno effect.

Zeno's original paradox:

In his original 'arrow paradox', zeno claimed that if one looks at an arrow in mid flight, at any given moment it appears to be standing still, so one must conclude that the arrow never moves.

The quantum Zeno effect:
The quantum zeno effect is generated by measuring a system at equal and short time spaces, inhibiting its decay. This is unintuitive, since we would expect that adding a perturbation to a system will cause it to decay more rapidly. This stems from the nature of quantum decay at short times and the collapse of wave functions into a definitive state with measurement.

For short times, the survival probability of a system goes approximately like t^2. We denote survival and decay probabilities as $P_s(t)$ and $P_d(t)$ respectively. We have

$$P_s(t) = 1 - P_d(t) = 1 - \sum_{n \neq n_0} |\langle n| e^{-iHt} |n_0\rangle|^2 \overset{t \to 0}{=} 1 - \sum_{n \neq n_0} |\langle n| 1 - iHt |n_0\rangle|^2 \tag{9}$$

Where the first term in the sum goes to 0 and we have:

$$P_s(t) = 1 - \sum |H_{n,n_0}|^2 t^2 = e^{-(\sum |H_{n,n_0}|^2) * t^2} \tag{10}$$

Thus, we see that a quantum system's decay in short times will be proportional to t^2.

If we denote $\sum_{n \neq n_0} |H_{n,n_0}|^2$ as $|V|^2$, for a measurement after time τ we have

$$P_s(\tau) = 1 - |V|^2\tau^2 \tag{11}$$

and for $n = t/\tau$ measurements we have

$$P_s^{(n)}(t) = (1 - |V|^2\tau^2)^{\frac{t}{\tau}} = e^{-(|V|^2\tau)t} \tag{12}$$

where $P_s^{(n)}(t)$ is the probability of finding the particle undecayed in each measurement (this is important). We get a decay factor of

$$\Gamma = |V|^2\tau \tag{13}$$

and it is also easy to see that for $n \to \infty$

$$P_s(t) \to 1 \tag{14}$$

Showing that the more frequent the measurements are (in the zeno region) the less the system will decay.

Probability matrix:
Next we will look at the system from the probability matrix point of view [2]. We would like to calculate $Q(\Delta, n, \rho)$ - The probability of measuring the system as undecayed (Initial state is ρ) for all measurements in the time space $\Delta = [0, t]$ with $t = \tau n$, where n is the number of measurements performed, every τ.

We denote $\rho(n, t)$ as the state of the system after n measurements (with $\Delta = [0, t]$). According to the theory of measurements, after an ideal measurement of observable E which yields true (meaning particle is undecayed) we have

$$\rho' = E\rho E \tag{15}$$

Between every measurement our system will evolve under $e^{-iHt/n}$, and the (unnormalized) state of the system will be:

$$\rho(n, t) = T_n^*(t)\rho T(t/n) \tag{16}$$

where

$$T_n(t) = [EU(t/n)E]^n = [Eexp(-iHt/n)E]^n \tag{17}$$

From here we can calculate $Q(\Delta, n, \rho)$ as:

$$Q(\Delta, n, \rho) = Tr([T_n(t)\rho T_n^*(t)]) \tag{18}$$

Now, if we take the limit $n \to \infty$ as the contiuous measurement limit, we have:

$$Q_{cont}(\Delta, \rho) = \lim_{n\to\infty} Q(\Delta, n, \rho) \tag{19}$$

and if the following limit exists

$$\lim_{n\to\infty} T_n(t) = \lim_{n\to\infty} [EU(t/n)E]^n = T(t) \tag{20}$$

we have:

$$Q(\Delta, \rho) = Tr([\rho T^*(t)T(t)]) \tag{21}$$

As the survival probability for continuous measurement.

QZE for the model two-site system:
Let us now consider a two site system with coupling Ω.
The Hamiltonian of the system is: $\hat{H} = \frac{1}{2}\Omega(|1\rangle\langle 2| + |2\rangle\langle 1|)$ (we assume for the sake of simplicity that the system is symmetric to reflections) and it performs Rabbi oscillations with frequency Ω.
The evolution operator is described as $U(t) = e^{-i\hat{H}t} = [cos(\frac{1}{2}\Omega t) - isin(\frac{1}{2}\Omega t)][|1\rangle\langle 2| + |2\rangle\langle 1|]$. Thus, the transition probability, in the symmetrical case where the system is prepared at state $|1\rangle$ is:

$$P_2(|1\rangle) = |\langle 1|U(\tau)|2\rangle|^2 = sin^2(\frac{1}{2}\Omega\tau) \tag{22}$$

and we can see that as $\Delta t \to 0$, $P_2(|1\rangle) \to 0$ in a quadratic pace. Calculating the decay factor from (10) we finally get:

$$\Gamma = \frac{\Omega^2\tau}{2} \tag{23}$$

Probability matrix approach:
For the same system we have

$$\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{24}$$

and

$$H = \begin{pmatrix} 0 & \Omega/2 \\ \Omega/2 & 0 \end{pmatrix} \tag{25}$$

With projector E

$$E = |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{26}$$

as the occupation of the ground state (initial occupancy level), giving

$$T_n = [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\frac{1}{2}\Omega\tau) & -\sin(\frac{1}{2}\Omega\tau) \\ -\sin(\frac{1}{2}\Omega\tau) & \cos(\frac{1}{2}\Omega\tau) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]^n = \begin{pmatrix} \cos^n(\frac{1}{2}\Omega\tau) & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

and finally

$$P_{surv}(t) = \cos^{2n}(\frac{1}{2}\Omega\tau) \approx (1 - \frac{1}{4}\Omega^2\tau^2)^{2n} \approx e^{-\frac{1}{2}\Omega^2\tau * t} \tag{28}$$

Giving, once again, and effective decay rate of

$$\Gamma = \frac{1}{2}\Omega^2\tau \tag{29}$$

as before.
Anti-Zeno Effect :
An acceleration of the decay of a quantum system can occur, at measurements of specific times, leading to the QAZE - an acceleration in decay of a system as a result of measurements (This can also happen with continuous measurements, as can be seen in [6]).

If we define an effective decay rate

$$\gamma_{eff}(\tau) = -\frac{n}{\tau}\ln(P_s^{(1)}(t)) \tag{30}$$

we can denote survival probability after n measurements at time spaces $\tau = t/n$ as

$$P^{(N)}(t) = e^{-\gamma_{eff}*t} \tag{31}$$

Now, for a system with exponential decay

$$P(t) = e^{-\Gamma t} \tag{32}$$

we can try to solve

$$\Gamma = \gamma_{eff}(\tau) \tag{33}$$

and find time τ_c for which these two probabilities cross. This time is the border between QZE and QAZE. We note that there isn't neccessarily a solution for (33), and the QAZE is not guaranteed. The above is demonstrated graphically in fig. 2, where τ_1 is the time taken for QZE and τ_2 is the time taken for QAZE.

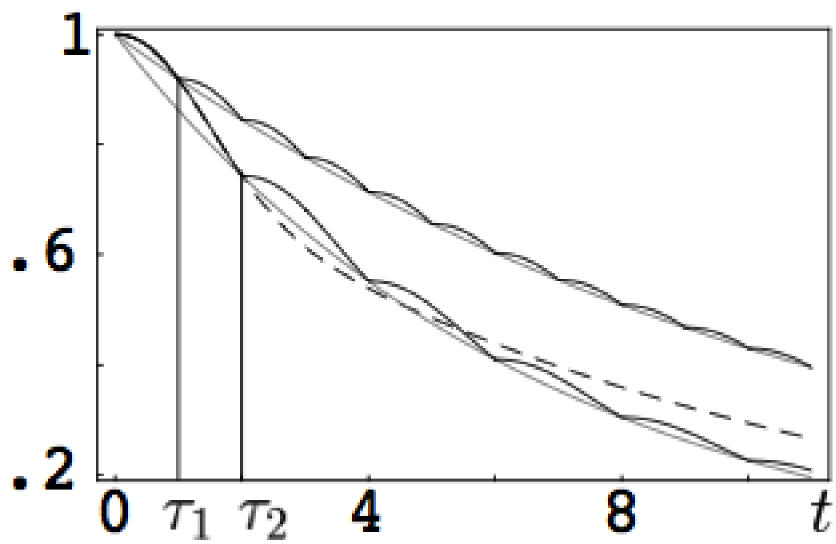

Fig. 2 [3] - QZE vs. QAZE. The dotted line is the system's "natural" evolution. Lines representing measurement in τ_1 and τ_2 time spaces correspond to the QZE and QAZE respectively. A "slowing down" of the evolution is seen for QZE and a "speeding up" for QAZE.

Raizen's experiment [4]:

Raizen showed in his experiment [4] the behavior of an unstable system under frequent and equally spaced measurements, displaying both QZE and Anti-QZE for a system of trapped sodium atoms with tunneling possibility.

The experiment involved ultracold sodium atoms under an accelerated wave potential of $V_0 \cos[2K_L x - K_L a t^2]$, where V_0 is the potential amplitude, K_L is the wave number, x is the position in the laboratory frame, a the acceleration and t the time.

Under manipulations, the experimental team were able to keep only the lowest energy band populated [4] (other bands population was negligible). Fig. 3 shows the band structure of the wave potential. The team then accelerated the standing wave modulation, creating Vannier-Stark ladders [5], allowing the atoms in the lowest energy band to tunnel out of the barrier. Fig. 4 shows the effective potential the particles feel under the acceleration. Then, by cooling the system down and taking snapshots of atom positions, the team were able to determine the concentration in space of the atoms. By measuring at short time spaces they were able to show the QZE. By increasing the time span between measurements they were able to show the QAZE.

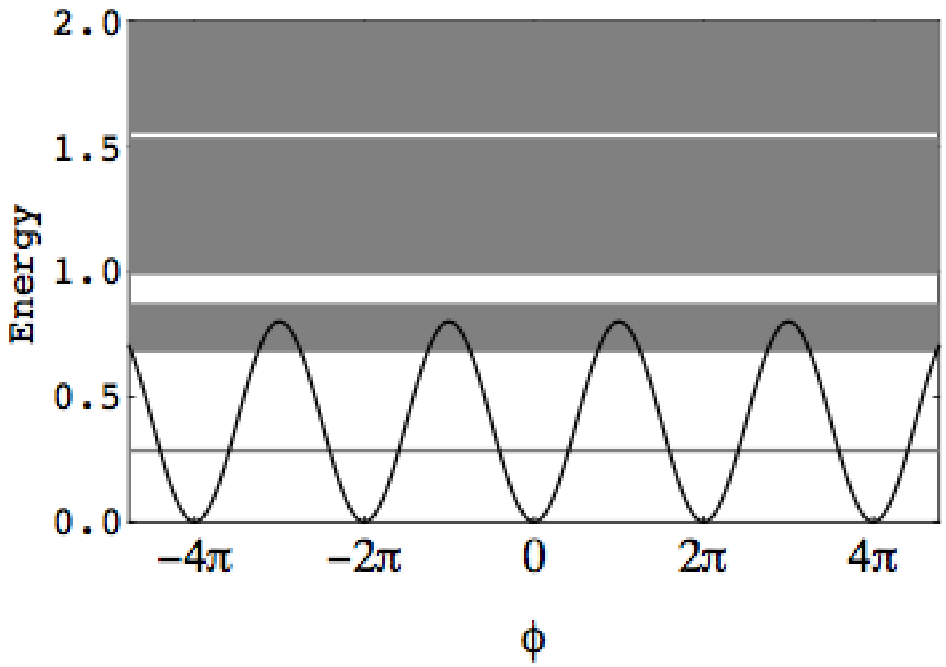

Fig. 3 [5] - The band structure of the optical lattice in Raizen's experiment. Full line is the potential plotted as a function of position $\phi = 2k_L x$. Gray area represents allowed energies, while white area the evergy gaps.

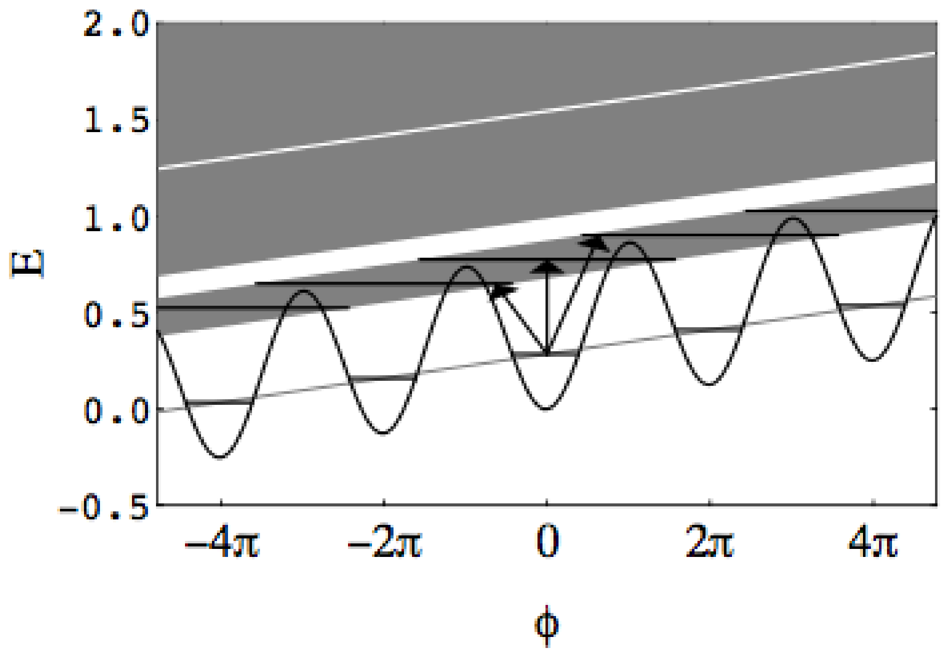

Fig. 4 [5] - Tilted bands and Wannier-Stark ladders in Raizen's experiment. Arrows indicate resonant excitations by an AC modulation of the acceleration. $\phi = 2k_L x$

The team were able to demonstrate, using different time spaces (about $1\mu s$ for QZE and $5\mu s$ for QAZE) the inhibition in decay and the accelerated decay from the QZE and QAZE respectively. Fig. 5 shows the results for QZE, whereas Fig. 6 shows the results for QAZE:

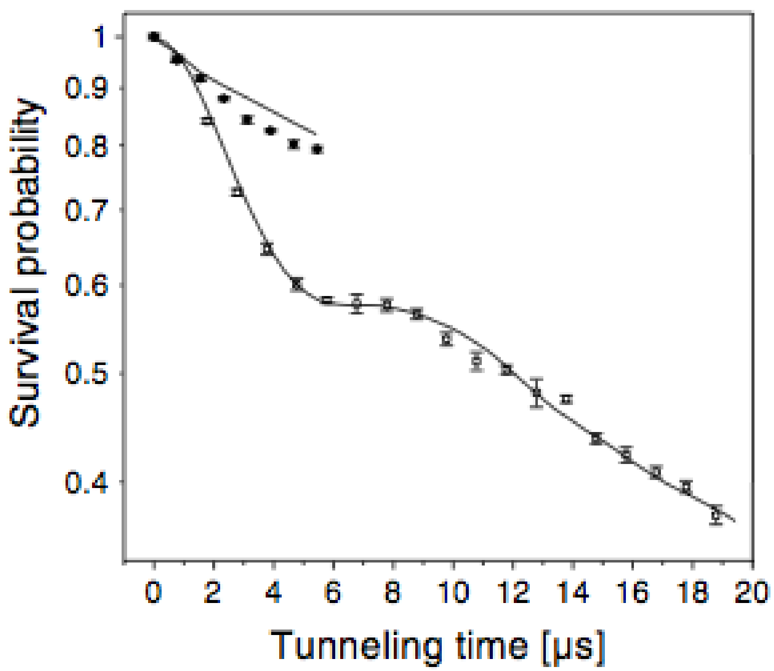

Fig. 5 [4]: QZE in the raizen experiment system. The hollow spheres represent the uninterrupted system's survival probability, whereas the full spheres represent the interrupted system's survival probability. A clear inhibition of decay is shown.

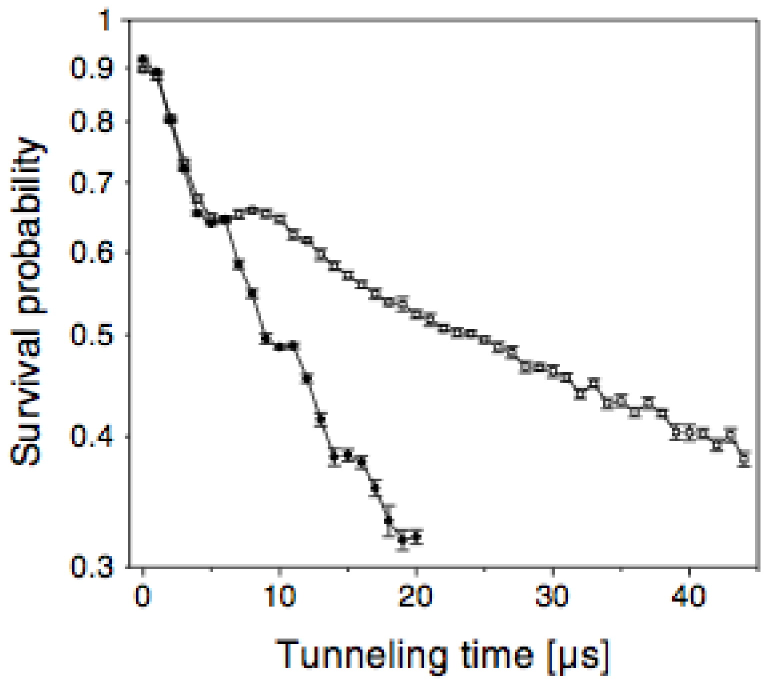

Fig. 6 [4]: QAZE in the raizen experiment system. The hollow spheres represent the uninterrupted system's survival probability, whereas the full spheres represent the interrupted system's survival probability. This time a clear acceleration of decay is shown.

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