Weak measurements

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[1] Pre and Post selection in strong measurements

Lets take a state:

$$|\Psi\rangle = \sum_n a_n |n\rangle. \tag{1}$$

The probability for a "collapse" to the particular state $\langle n|$ is given by the projection formula:

$$Prob(n|\Psi) = |\langle n|\Psi\rangle|^2 \tag{2}$$

Now let us take an operator $\hat{A}$ that its result when measuring the state $|n\rangle$ is:

$$\hat{A}|n\rangle = a_n|n\rangle \tag{3}$$

then the expectation value $\langle \hat{A}\rangle$ of $\hat{A}$ with respect to the state $|\Psi\rangle$ is defined by:

$$\langle \hat{A}\rangle_{\Psi}^{strong} \equiv \frac{\sum_n Prob(n|\Psi)\ a_n}{\sum_n Prob(n|\Psi)} = \frac{\langle\Psi|\hat{A}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \xrightarrow[states]{normalized} \langle\Psi|\hat{A}|\Psi\rangle \tag{4}$$

Let us calculate what is the probability of (2) to finish at state $|\Phi\rangle$. We use the same method:

$$Prob(\Phi|n|\Psi) \equiv Prob(\Phi|n)Prob(n|\Psi) = |\langle\Phi|n\rangle|^2\ |\langle n|\Psi\rangle|^2 \tag{5}$$

So we get that the expectation value now is:

$$\langle \hat{A}\rangle_{\langle\Phi|\Psi\rangle}^{strong} = \frac{\sum_n |\langle\Phi|P^{(n)}|\Psi\rangle|^2 a_n}{\sum_n |\langle\Phi|P^{(n)}|\Psi\rangle|^2} \tag{6}$$

Where:

$$P^{(n)} \equiv |\Psi_n\rangle\langle\Psi_n| \ \text{ is the "projection operator".} \tag{7}$$

[2] The three boxes paradox

Let us consider a particle that can be located in one of three orthogonal quantum boxes. We prepare the particle at time t_{past} in an equally weighted superposition state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \tag{8}$$

We notice that every eigenstate has the same collapse probability.

Next let us assume that at a later time t_{future} the particle is found to be in the state:

$$|\Phi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle) \tag{9}$$

If we preform a strong measurement of an observable A which has degenerate eigenstates and we want to know the probability for the outcome a_n eigenvalue, we can use:

$$Prob(A = a_n) = \frac{|\langle\Phi|P^{(n)}|\Psi\rangle|^2}{\sum_i |\langle\Phi|P^{(i)}|\Psi\rangle|^2} \tag{10}$$

This is called the ABL rule [1,2].

We'll calculate the probability of finding the particle in box 1:

$$P(P_1 = 1) = \frac{|\langle\Phi|1\rangle\langle1|\Psi\rangle|^2}{|\langle\Phi|1\rangle\langle1|\Psi\rangle|^2 + |\langle\Phi|2\rangle\langle2|\Psi\rangle + \langle\Phi|3\rangle\langle3|\Psi\rangle|^2} = \frac{|\frac{1}{3}|^2}{|\frac{1}{3}|^2 + |0|^2} = 1 \tag{11}$$

So if we open box 1 we will definitely fine the particle there. The same goes for box 2.

Lets check out box 3:

$$P(P_3 = 1) = \frac{|\langle\Phi|3\rangle\langle3|\Psi\rangle|^2}{|\langle\Phi|3\rangle\langle3|\Psi\rangle|^2 + |\langle\Phi|1\rangle\langle1|\Psi\rangle + \langle\Phi|2\rangle\langle2|\Psi\rangle|^2} = \frac{|\frac{1}{3}|^2}{|\frac{1}{3}|^2 + |\frac{2}{3}|^2} = \frac{1}{5} \tag{12}$$

We can summarize this case as follows: in classical physics the particle has only one position therefore only opening one specific box will corresponds with the result finding the particle in the box with a probability of $\frac{1}{N}$. But a quantum particle acts differently since it does't matter which box of the N-1 (generalized case) boxes we open (always) we'll find the single particle there for sure. That is why we say that the particle is simultaneously in N-1 boxes. Finely we deduce that as more boxes there are the chance we'll find it in the last box decreases.

A different way to emphasize this result is to let Alice prepare the pre selected state as (8) and let her measure the post selected state. If Bob sees the particle at 1 or 2 (he chooses a box) then if the post selected state is like (9) Alice knows that he saw it. If Bob doesn't see the particle for example he measured 1 then the particle at time t is at the state $\frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ and that is what Alice will measure and because it's different from (9) she'll know he didn't see the particle for sure.

═══ [3] Von Neumann with post selection and Weak measurements

Von Neumann was first to proposed a theory modelling a measurement through interaction. Its principle is to obtain the value of a measured observable through the observable of measurement device such as a shift of a pointer variable during the interaction.

In an ideal measurement the initial position of the pointer q is well localized around the zero and therefore the conjugate momentum $\hat{p}$ has a very large uncertainty. In a weak measurement the initial state of the measuring device is such that $\hat{p}$ is localized around zero with small uncertainty so there is a large uncertainty in q and therefore the measurement becomes inaccurate. So in contrast to the classic Von Neumann approach of measuring quantum states, weak measurement uses a measuring device whose pointer has a very large uncertainty when compared with the regular shift.

In Ref [3] it has been suggested to define $\langle A\rangle_{\Psi\Phi}^{weak} = \frac{\langle\Phi|A|\Psi\rangle}{\langle\Phi|\Psi\rangle}$. We would like to interpret this result within the framework of the Von-Neumann measurement scheme which is used also in the study of weak continuous measurements [4-8]. In particular we would like to clarify the significance of the imaginary part.

The coupling of the system to a Von-Nuemann pointer whose canonical coordinates are $(\hat{x}, \hat{q})$ is described by the Hamiltonian:

$$H = -\lambda g(t) A\hat{x} \tag{13}$$

where $g(t)$ is a short time normalized function, λ is assumes to be very small and A is the observable that is a measured Hermitian operator.

In general A is an observable that has a spectrum of values $\{a_i\}$ so that if $A = a$ this interaction shifts the pointer:

$$q \mapsto q + \lambda a \tag{14}$$

Note that x unlike q is a constant of the motion.

For a weak measurement we assume that λ is small and obtain the following result: the evolution of the system with the pointer is described by:

$$U = \mathrm{e}^{-i\int H dt} = \mathrm{e}^{-i\int -\lambda g(t) A\hat{x} dt} = \mathrm{e}^{i\lambda A\hat{x}} \approx \quad 1 + i\lambda A\hat{x} \tag{15}$$

The matrix elements of this evolution operator are:

$$\langle\Phi, x|U|\Psi, x_0\rangle \quad = \quad U[x]_{\Phi,\Psi}\ \delta(x - x_0) \tag{16}$$

where $U[x]$ is a system operator that depends on the constant parameter x.

If the system is prepared in the state $P^\Psi = |\Psi\rangle\langle\Psi|$, and the pointer is prepared in state $\rho^{(0)}$, then after the interaction we get:

$$\text{state} = UP^\Psi\rho^{(0)}U^\dagger \tag{17}$$

The reduced state of the pointer after post selection is

$$\rho(x'', x') \quad = \quad \text{trace}\Big[\ P^\Phi Q^{x'',x'}\ UP^\Psi\rho^{(0)}U^\dagger\ \Big] \tag{18}$$

where $Q^{x'',x'} = |x'\rangle\langle x''|$. Note that the state is not normalized: the trace is the probability to find the system in state Φ.

The relation between ρ and $\rho^{(0)}$ can be written as follows:

$$\rho(x'', x') \quad = \quad \tilde{K}(x'', x')\ \rho^{(0)}(x'', x') \tag{19}$$

where:

$$\tilde{K}(x'', x') \quad = \quad \langle\Phi|U[x']|\Psi\rangle\ \langle\Psi|U[x'']^\dagger|\Phi\rangle \tag{20}$$

Equivalently, we can transform to the Wigner representation by changing variables and then multiplication becomes a convolution:

$$\rho(X,r) = \tilde{K}(X,r)\ \rho^{(0)}(X,r) \tag{21}$$

$$\rho(X,q) = \int K(X,q-q')\ \rho^{(0)}(X,q')\ dq'$$

where

$$K(X,q-q') = \int \tilde{K}(X,r)\ \mathrm{e}^{-i(q-q')r} dr \tag{22}$$

$$\tilde{K}(X,r) = \left\langle \Psi \middle| U\left[X-\tfrac{r}{2}\right]^{\dagger} \middle| \Phi \right\rangle \left\langle \Phi \middle| U\left[X+\tfrac{r}{2}\right] \middle| \Psi \right\rangle$$

If we sum over Φ we get the standard result for weak measurement without post-selection. We receive:

$$\langle\Psi|(1-i\lambda(X-\tfrac{r}{2})A^{\dagger})(1+i\lambda(X+\tfrac{r}{2})A)|\Psi\rangle \tag{23}$$

$$= \langle\Psi|(1+i\lambda X(A-A^{\dagger})+i\lambda\tfrac{r}{2}(A+A^{\dagger}))|\Psi\rangle$$

and because A is a hermitian operator meaning $A^{\dagger}=A$ and $\langle\Psi|\hat{A}|\Psi\rangle = \mathrm{Re}\langle A\rangle$ we get that (23) is equal to:

$$\mathrm{e}^{i\lambda r \mathrm{Re}\langle A\rangle} \tag{24}$$

So finally we can see that (22) becomes:

$$K(X,q-q') = \int dr \mathrm{e}^{-ir(q-q'-\lambda\mathrm{Re}\langle A\rangle)} = 2\pi\delta(q-q'-\lambda\mathrm{Re}\langle A\rangle) \tag{25}$$

So indeed when $\lambda=1$ for strong measurement (14) is correct producing an eigenstate.

Now let us go back and assume there is a post selected state:

$$\langle\Psi|(1-i\lambda(X-\tfrac{r}{2})A^{\dagger}|\Phi\rangle\ \langle\Phi|(1+i\lambda(X+\tfrac{r}{2})A|\Psi\rangle \tag{26}$$

$$= \langle\Psi|\Phi\rangle\Big(1-i\lambda(X-\frac{r}{2})\frac{\langle\Psi|A^{\dagger}|\Phi\rangle}{\langle\Psi|\Phi\rangle}\Big)\langle\Phi|\Psi\rangle\Big(1+i\lambda(X+\frac{r}{2})\frac{\langle\Phi|A|\Psi\rangle}{\langle\Psi|\Phi\rangle}\Big)$$

We define:

$$\frac{\langle\Psi|A^{\dagger}|\Phi\rangle}{\langle\Psi|\Phi\rangle} \equiv A_1\ ,\ \frac{\langle\Phi|A|\Psi\rangle}{\langle\Phi|\Psi\rangle} \equiv A_2 \tag{27}$$

So we get that (26) is:

$$1+i\lambda X(A_2-A_1)+i\lambda\tfrac{r}{2}(A_2+A_1) \tag{28}$$

We notice that A is a complex number: $A = \mathrm{Re}\langle A\rangle + i\mathrm{Im}\langle A\rangle$ and because A is hermitian $A_1 = A_2^{\dagger}$ so (28) is:

$$1+i\lambda(X2i\mathrm{Im}\langle A_2\rangle + r\mathrm{Re}\langle A_2\rangle) = \mathrm{e}^{i\lambda(2Xi\mathrm{Im}\langle A_2\rangle + r\mathrm{Re}\langle A_2\rangle)} \tag{29}$$

So (22) this time becomes:

$$K(X,q-q') = \mathrm{e}^{-2\lambda\mathrm{Im}\langle A_2\rangle X}\ 2\pi\delta(q-q'-\lambda\mathrm{Re}\langle A_2\rangle) \tag{30}$$

where $\langle A\rangle$ is define as the "weak value" by Yakir Aharonov.

This means that if one starts with a minimal Gaussian of width σ, its center will be shifted as follows:

$$\text{q-shift} = \text{Re}[\langle A\rangle^{\text{weak}}_{\Phi,\Psi}](\lambda) \tag{31}$$
$$\text{x-shift} = \text{Im}[\langle A\rangle^{\text{weak}}_{\Phi,\Psi}](-2\sigma^2)$$

Now we can see once again that a weak measurement that doesn't have a post selected state is like a regular strong measurement:

$$\langle A\rangle^{weak}_{\Psi} = \sum_{\alpha} Prob(\alpha) A^{weak}_{\Phi^\alpha|\Psi} = \sum_{\alpha} |\langle\Phi^\alpha|\Psi\rangle|^2 \frac{\langle\Psi^\alpha|A|\Psi\rangle}{\langle\Phi^\alpha|\Psi\rangle} = \sum_{\alpha}\langle\Phi^\alpha|\Psi\rangle^*\langle\Phi^\alpha|A|\Psi\rangle = \langle\Psi|A|\Psi\rangle \tag{32}$$

Which is the known result as we saw at (4).

====== [4] Experiment preforming weak measurements

Stage1 is preparing the pre selected state: lets take a beam of spin $\frac{1}{2}$ particles that moves in the y direction. The particles' spins point in the $x-z$ plane in the direction $\hat{n}$ which makes an angle θ with the z axis.

Stage2 is to measure it weakly: the beam passes through a Stern-Gerlach device that measures the spin weakly in the z direction. It succeeds doing so because its gradient of the magnetic field is sufficiently small. Therefore the motion of the beam changes only slightly.

Stage3 is having the post selected state: finally we pass the beam through another S-G device this time which has a normal gradient. Its job is to split the beam into two beams corresponding to the two values of σ_x. We take the particles with $\sigma_x = +1$ and the beam finally hits a screen. On the screen we obtain a wide spot whose displacement in the direction $\hat{z}$ is measured. This displacement will yield the weak value of σ_z.

The initial spin state $|\Psi\rangle$ is:

$$|\Psi\rangle = cos(\tfrac{\theta}{2})|\uparrow\rangle + sin(\tfrac{\theta}{2})|\downarrow\rangle \tag{33}$$

and the final state $|\Phi\rangle$ is:

$$|\Phi\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{34}$$

So we get that:

$$\langle\Phi|\Psi\rangle = \tfrac{1}{\sqrt{2}}(cos(\tfrac{\theta}{2}) + sin(\tfrac{\theta}{2}))\ ,\ \langle\Phi|\hat{\sigma}_z|\Psi\rangle = \tfrac{1}{\sqrt{2}}(cos(\tfrac{\theta}{2}) - sin(\tfrac{\theta}{2})) \tag{35}$$

We take the weak value of the spin component σ_z. This displacement will yield the weak value of the spin component σ_z:

$$\langle\hat{\sigma}_z\rangle^{weak}_{\langle\Phi|\Psi\rangle} = \frac{\langle\Phi|\sigma_z|\Psi\rangle}{\langle\Phi|\Psi\rangle} = \frac{cos(\theta)}{1 + sin(\theta)} \tag{36}$$

Substituting this in (30) shows us the movement of the pointer. We notice that if $\theta \to -\pi$ we get very large values that exceeds the eigenvalue spectra of spin $\frac{1}{2}$.

Appendix: Philosophical discussion on the TSVF

——— [1] Description of a quantum system in the standard approach

In quantum mechanics we denote a system at a given time t_{now} by a quantum state:

$$|\Psi\rangle \tag{1}$$

This state is defined by the result of a measurement that was performed at the past meaning $t_{measure} < t_{now}$. The quantum state yields maximal information about how the system can affect the environment i.e. interacting with measuring devices at time t_{now}.

The result of the interaction can't be exactly determined ahead a time i.e. the position of an electron during the "collapse" in the two slit experiment. Quantum mechanics is a deterministic probability distribution theory; the interaction of the same observable acting on the same environment can bring different results. But we can quantify this supposedly arbitrariness: in quantum mechanics we define for every observable a hermitian operator for which we can define an expectation value, which gives us the average of N results, where N goes to infinite, of the same measurements:

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle \tag{2}$$

In classical physics by knowing all the parameters of a system at time t_{now} by measuring them at time $t_{measure}$ we can know how the state of the system will develop in time. By flipping the direction of the time arrow the results of measurements in the past are defined by the results of measurements in the future. Therefore classic mechanics is time symmetric.

Controversially in quantum mechanics this is not true: the results of measurements in the future are not completely constrained by the results of measurements in the past therefore the concept of a quantum state is time asymmetric.

——— [2] TSVF, pre and post selection

A system at a given time t_{now} can be described as we saw by eq. (1). But it can also be described by the Two State Vector Formalism - TSVF as fallows:

$$\langle \Phi | \quad | \Psi \rangle \tag{3}$$

This method pioneered by Aharonov, Bergmann and Lebowits - ABL in short, provides a time-symmetric formulation of quantum mechanics instead of the asymmetry formulation.

We define a time t which fulfils the relation:

$$t_{measure} = t_{past} < t < t_{future} < t_{now} \tag{4}$$

Like in (1) the state $|\Psi\rangle$ is defined by the result of a measurement performed on the system at time t_{past}. Now we add the state $\langle \Phi |$ defined by the result of a measurement performed on the system at time t_{future}.

Contradict to the standard approach the TSVF adds the constricting state $\langle \Phi |$ giving the result of a measurement in the future and by that really maximals information for the system at time t. The idea of this concept is to squeeze more information on a system that has existed in the past than the standard approach. There is no meaning of determining $t = t_{now}$; we are dealing with 3 points in time which are all in the past comparing to t_{now}.

We call the state $|\Psi\rangle$ the "pre-selected state" which is the state we prepare the system at and we call the state $\langle\Phi|$ the "post-selected state" which is the state the system is at the end of the process. These two measurements are strong measurements.

We notice that similarly to eq. (1) formalism the TSVF yields maximal information about how the system can affect the environment when interacting with it at time t and that both have the same predictions for the system. The first questions that arises immediately is: Then how is it different? The philosophical side of the answer is that we want to understand the true interpretation of nature and in this case its symmetric side of quantum mechanics. The practical side is that it available us to use the "weak measurements" method which will be discuss.

====== [3] Interpretation of the TSVF

There are all kinds of interpretations for such a case. One of the more dramatic ones is that supposedly because the state at t_{future} determines the state at time t than the future determines the past. This is an interpretation that helps to understand the arbitrariness feature of the "collapse" because we can deduce that a repeated experiment meaning the same measurement of the same quantum state gives different results because of the backward developing state in time, which we do not know, that is different so actually it is not the same experiment.

Let us consider the next classical analogy: a kid receives to his hands at time $t_{measure}$ marbles with 3 different features: size [big or small], colour [black or white] and weight [heavy or light]. Let us assume that at time $t_{measure}$ he takes out and throws away all the black and heavy marbles and calls the marbles that are left "the state $|\Psi\rangle$". Now let us assume that in time t_{final} he takes all the small and white marbles and throws them away and calls the marbles that are left "the state $\langle\Phi|$". Immediately after time t_{final} he leaves the marbles that are left on the ground and goes home. For someone that hears the first part of the story there are much more types of "marble states" that can exist at the future at time t_{final} but if someone would tell him what the kid had in mind he would get a better and a more realistic picture on the options left in the future.

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