

# Tunneling via a barrier faster than light

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Numerous theories contradict to each other in their predictions for "the" tunneling time.

## 1 The Wigner time delay

Consider particle which described by following Gaussian wave packet is incident on a potential barrier:

$$\phi_I(x, t) = \int_0^\infty dk e^{-\sigma(k-k_0)^2} \exp [i(k(x - x_0) - E(k)t)] \quad (1)$$

Thus each particular wave component of transmitted wave packet is multiplied by some factor  $D \equiv \sqrt{g}e^{i\theta}$  and has following form (where  $g$  is a transmission coefficient or transmission probability):

$$\phi_T(x, t) = \int_0^\infty dk e^{-\sigma(k-k_0)^2} \sqrt{g} \exp [i(k(x - x_0) + \theta - Et)] \quad (2)$$

Thus motion of wave packet peak on the right from the barrier can be described by

$$\bar{x} = x_0 + v_g(t - \tau_w) \quad (3)$$

Where

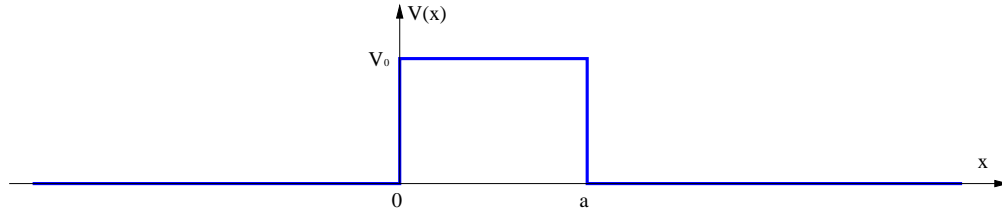
$$\tau_w = \hbar \frac{d\theta}{dE} \quad (4)$$

$$v_g = dE/dk \quad (5)$$

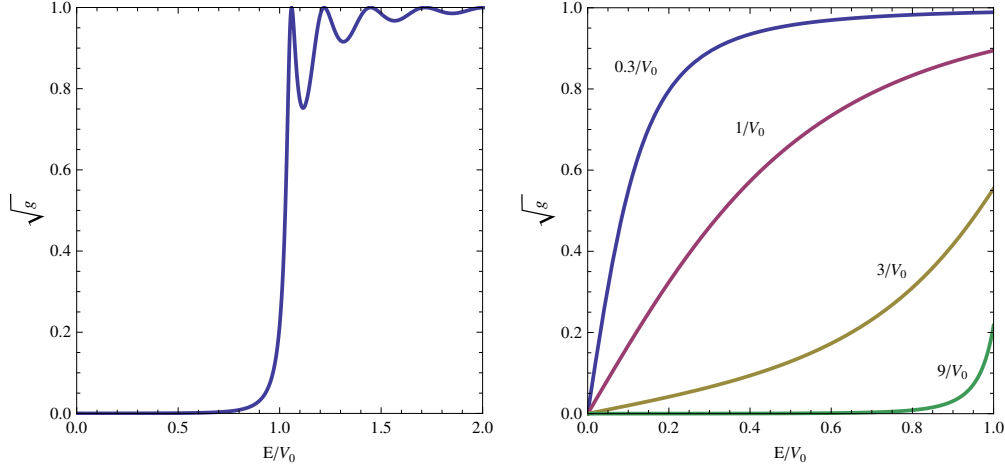
$\tau_w$  got name of Wigner time delay and equals to 0 in the absence of the barrier. At  $t = 0$  wave packet peak was at some  $x_0$  on the left from the barrier. Passage through the barrier gave effect of time delay  $\tau_w$  and peak continued its classical motion which is characterized by group velocity  $v_g$ .

### 1.1 Hartman effect

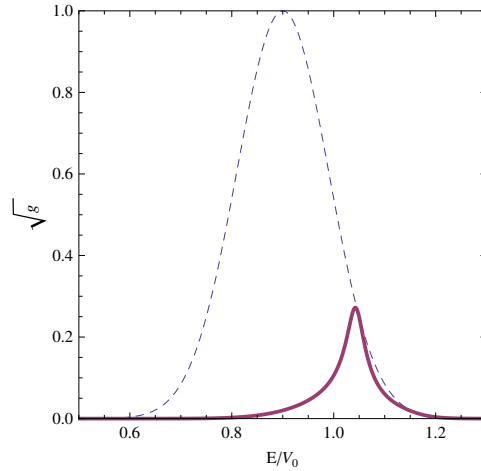
Let us consider the case when particle is incident on rectangular barrier from the left.



Square root of transmission coefficient is plotted as a function of incident wave energy in following figure on the left (in units of barrier height) and zoom in of transmission coefficient in a under barrier region for different widths of barrier is shown on the right figure:



For very thick barriers ( $a$  is a barrier width) most of the tunneling takes place very near the top of the barrier. Thus only high energies succeed to pass through the barrier and the peak of transmitted wave packet is shifted to somewhat higher energy as shown on figure below (dash line is the original (incident) wave packet).



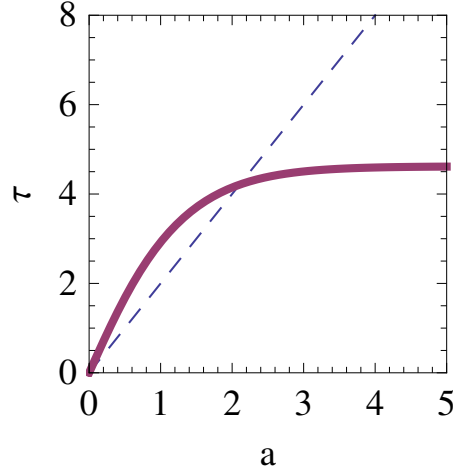
We define tunneling time in sense of Wigner delay

$$\tau = \tau_0 + \tau_w \tag{6}$$

Where  $\tau_0 \equiv a/v_g$  is called "equal time", the time which takes to wave packet peak to travel a distance equal to barrier width in free region.

Hartman found that for thin barriers the transmitted packet has essentially the same form as the incident packet and that its  $\tau$  is greater than the equal time. For thicker barriers the peak of the transmitted packet is shifted to somewhat higher energies as a result of the filtering action of the barrier. Most importantly, in what has become known as the Hartman effect, he found that  $\tau$  becomes independent of barrier thickness and is shorter than the equal time. For very thick barriers the below-barrier components are suppressed so that the propagating above-barrier components begin to dominate, at which point the delay time begins to increase again (classical region).

Following graph shows how  $\tau$  depends on barrier width (dashed line is a  $\tau_0$ ). For the thick barriers tunneling time  $\tau$  is approaching upper limit which is given by (8).



Additional explanation why for thick barrier time delay stays unchanged can be achieved from WKB approximation for square root of transmission coefficient which gives good approximation for slow-varying potential and for  $x$  far enough from turning points  $x_1$  and  $x_2$ .

$$\sqrt{g} = \exp \left[ - \int_{x_1}^{x_2} \sqrt{2m(V(x) - E)} dx \right] \quad (7)$$

The reason for time delay is that wave function gains the phase across the barrier. In above expression only magnitude of transmission decreases as long as barrier becomes thicker, but no phase is being gained. Tunneling time upper limit for thick barrier according to Hartmat is given by

$$\tau = \frac{2m}{\hbar k} \frac{1}{\alpha} \quad (8)$$

Where  $k \equiv \sqrt{2mE}$  and  $\alpha \equiv \sqrt{2m(V(x) - E)}$ .

Effectively  $\ell \equiv 2/\alpha$  represented penetration depth and  $v_E = \hbar k/m$  is velocity of the incident particle, therefore we can rewrite last expression in form

$$\tau = \ell/v_E \quad (9)$$

Note that penetration depth and incident particle velocity don't depend on barrier width in case of thick barrier.

## 1.2 Dwell time of Smith

The dwell time was first introduced by Smith in a potential scattering context. He defined it as the difference between the time spent by a particle in the region of the scattering potential and the time spent in the same region in the absence of the scattering potential. In the context of tunneling, the dwell time was written down by Buttiker <sup>1</sup> as

$$\tau_d = \frac{\int_0^L |\psi|^2 dx}{J_{\text{in}}} \quad (10)$$

where  $J_{\text{in}} = \hbar k/m$  is incident flux and  $\psi(x)$  is the stationary state wave function. For example, for "very thick" rectangular barrier (infinite step) exact solution is

$$\tau_d = \frac{2m}{\hbar} \frac{k}{\alpha(k^2 + \alpha^2)} (1 - e^{-2\alpha a}) \approx \frac{2m}{\hbar} \frac{k}{\alpha(k^2 + \alpha^2)} \quad (11)$$

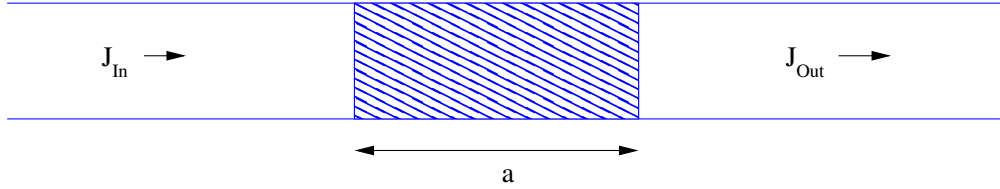
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<sup>1</sup>Don't be confused it's not Buttiker time.

WKB approximate solution for "very thick" rectangular barrier (infinite step) is

$$\tau_d = \frac{m}{2\hbar} \frac{1}{\alpha^2} (1 - e^{-2\alpha a}) \approx \frac{m}{2\hbar} \frac{1}{\alpha^2} \quad (12)$$

Intuitive explanation can be provided by following example. Consider liquid flowing through the tube. There is a region of length  $a$  inside the tube with some "distortion" which influences liquid speed and density as shown on figure



Continuity equation tell us that liquid flux should be the same via any cross section along the tube. Therefore we can write

$$J = \rho_0 v_0 = \rho_a v_a \Rightarrow v_a = \frac{J_{in}}{\rho_a} \quad (13)$$

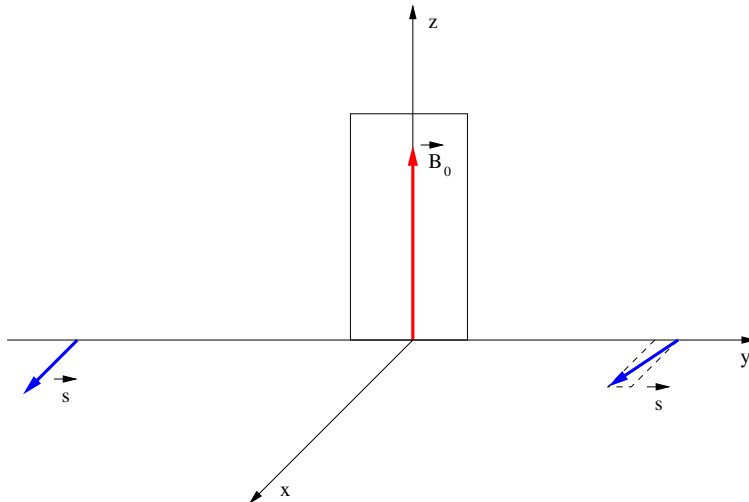
Where  $J$  is a liquid flux,  $\rho_0$  and  $v_0$  are liquid density and liquid velocity, respectively, before it enters the "distortion" region.  $\rho_a$  and  $v_a$  are liquid density and liquid velocity, respectively, inside the "distortion" region. Thus the time which particle spends in "distortion" region is

$$\tau_d = a/v_a = \frac{\rho_a a}{J_{in}} \quad (14)$$

It can be easily seen that equation (14) is a classical equivalent of definition (10). The dwell time does not distinguish transmitted particles from reflected particles since it is a property of an entire wave function with forward and backward components. If number of reflected particles much larger that number of transmitted particles dwell time doesn't reflect traverse time.

## 2 Traversal time of M.Buttiker and R.Landauer

In 1966, Baz' proposed the use of the Larmor precession as a clock to measure the duration of quantum-mechanical collision events. Rybachenko applied this method to the simpler case of particles in one dimension scattered at a barrier. This case we treat in this chapter. We consider particles with mass  $m$  and kinetic energy  $E = \hbar^2 k^2 / 2m$  moving along the  $y$  axis and interacting with a rectangular barrier of height  $V_0$  and width  $a$ , centered at  $y = 0$ . The particle carry spin  $1/2$  and the incident particles polarized in the  $x$  direction.



A small magnetic field  $\vec{B}_0$ , pointing in the  $z$  direction, is confined to the barrier. As particles enter the barrier they start a Larmor precession with frequency  $\omega_L = g\mu B_0/\hbar$ . Here,  $g$  is the gyro-magnetic ratio and  $\mu$  the absolute value of the magnetic moment. When the barrier leaves the barrier precession stops. The polarization of the transmitted particles is compared with the polarization of the incident particles. The angle between the initial and final polarization perpendicular to the field is assumed to be given by the Larmor frequency multiplied with the time a particle spends in the barrier. For this case Rybachenko found for the spin of transmitted particles to lowest order in the field  $B_0$ ,

$$\langle S_x \rangle \approx \hbar/2 \quad (15)$$

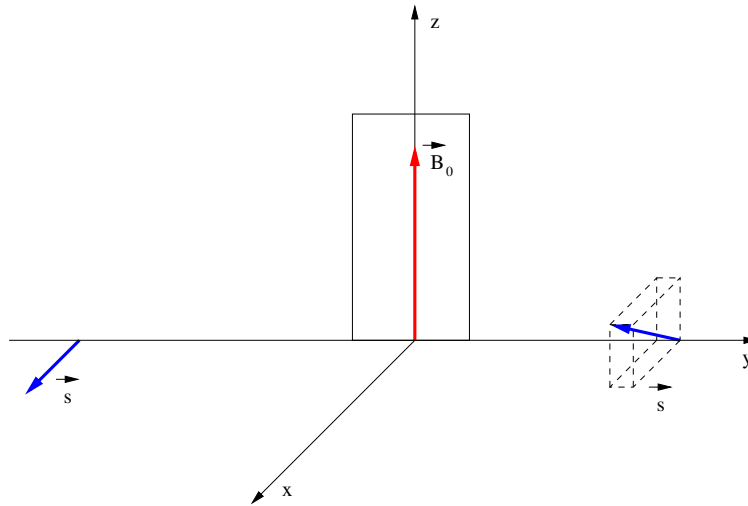
$$\langle S_y \rangle \approx -(\hbar/2)\omega_L\tau_y \quad (16)$$

Where we have introduced the time

$$\tau_y = \hbar k/V_0\alpha \quad (17)$$

The precession angle in  $x$ - $y$  plane is  $\omega_L\tau_y$  and Rybachenko concludes that  $\tau_y$  is the time a particle takes to traverse the barrier. Since equations (15) and (16) also holds for reflected particles Rybachenko concludes that time a reflected particle spends in the barrier is equal to the time a transmitted particle spends in the barrier.

A particle tunneling through a barrier doesn't actually perform a Larmor precession. The main effect of the magnetic field is to align the spin with a field. Thus incident particles polarized in the  $x$  direction will acquire a polarization component parallel to the field while tunneling through the barrier as shown on following figure



That particles polarized in the  $x$  direction acquire a  $z$  component when tunneling through the barrier can be understood in the following way. A beam of particles polarized in the  $x$  direction can be represented as a mixture of particles which have a  $z$  component  $\hbar/2$  with probability  $1/2$  and a  $z$  component  $-\hbar/2$  with probability  $1/2$ . Outside the barrier particles have kinetic energy  $E$  independent of the spin. But in the barrier the kinetic energy differs by the Zeeman contribution  $\pm\hbar\omega_L/2$ , giving rise to a different exponential decay for the wave functions within the barrier. Since transmission probability, in the absence of a field, is given by

$$T = \frac{16k^2\alpha^2}{(k^2 + \alpha^2)^2} e^{-2\alpha a} \quad (18)$$

particle with spin  $\hbar/2$  will penetrate the barrier more easily than particles with spin  $-\hbar/2$ . Neglecting corrections in the preexponential factors, this yields a transmission probability  $T_{\pm} = Te^{\pm\omega_L\tau_z}$ , where

$$\tau_z = ma/\hbar\alpha \quad (19)$$

## 2.1 Solution of Schrodinger equation for Buttiker problem

By exact solution of problem described above polarization of transmitted particles can be obtained

$$\langle S_x \rangle = \frac{\hbar}{2} \frac{g_+ - g_-}{g_+ + g_-} \quad (20)$$

$$\langle S_y \rangle = -\hbar \sin(\theta_+ - \theta_-) \frac{(g_+ g_-)^{1/2}}{g_+ + g_-} \quad (21)$$

$$\langle S_z \rangle = \hbar \cos(\theta_+ - \theta_-) \frac{(g_+ g_-)^{1/2}}{g_+ + g_-} \quad (22)$$

Where  $\theta_{\pm}$  is the accumulated phase difference for particle with spin  $\pm\hbar/2$  and it determined by

$$\tan(\theta_{\pm} + ka) = \frac{k^2 - \alpha_{\pm}^2}{2k\alpha_{\pm}} \tanh(\alpha_{\pm}a) \quad (23)$$

$g$  is transmission probability for particle with spin  $\pm\hbar/2$

$$g_{\pm} = \left( 1 + \frac{(k^2 + \alpha_{\pm}^2)^2}{4k^2\alpha_{\pm}^2} \sinh^2(\alpha a) \right)^{-1} \quad (24)$$

By taking field to be infinitesimally small above equations are reduced to the form

$$\langle S_z \rangle = \frac{\hbar}{2} \omega_L \tau_z \quad (25)$$

$$\langle S_y \rangle = -\frac{\hbar}{2} \omega_L \tau_y \quad (26)$$

$$\langle S_x \rangle = \frac{\hbar}{2} (1 - \omega_L^2 \tau_x^2 / 2) \quad (27)$$

Where  $\tau_y$  and  $\tau_z$  are central results of Buttiker:

$$\tau_z = -\frac{m}{\hbar\alpha} \frac{\partial \ln \sqrt{g}}{\partial \alpha} = \frac{mk_0^2 (\alpha^2 - k^2) \sinh^2(\alpha a) + \frac{1}{2}\alpha a k_0^2 \sinh(2\alpha a)}{\hbar\alpha^2 (4\alpha^2 k^2 + k_0^4 \sinh^2(\alpha a))} \quad (28)$$

$$\tau_y = \frac{m}{\hbar\alpha} \frac{\partial \theta}{\partial \alpha} = \frac{mk_0^2 2\alpha a (\alpha^2 - k^2) + k_0^2 \sinh(2\alpha a)}{\hbar\alpha (4\alpha^2 k^2 + k_0^4 \sinh^2(\alpha a))} \quad (29)$$

Where  $k_0 \equiv \sqrt{2mV_0}/\hbar$ .

Since

$$\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 = \hbar^2 / 4 \quad (30)$$

we must have

$$\tau_T = \tau_x = (\tau_y^2 + \tau_z^2)^{1/2} \quad (31)$$

Where  $\tau_T$  is interaction time of **transmitted** particle or **traversal time** of Buttiker.

See graph below which compares between Buttiker traversal time, dwell time and Wigner time delay.

## 2.2 Calculation of the phase-delay time for Buttiker problem

Substituting (23) into definition (4) we can calculate Wigner time delay in Buttiker problem. For small  $k$  it is given by

$$\tau_w = (2m/\hbar k \alpha) \tanh \alpha a \quad (32)$$

This result diverges as the kinetic energy of the incident particle tends to zero. This is in contrast to characteristic times  $\tau_z, \tau_y, \tau_x$  which all remain finite as  $k$  tends to zero. Also note that for an opaque barrier, the time  $\tau_w$  becomes independent of the width  $a$  of the barrier. It reminds us Hartman effect. The strong deformation of a wave packet when it interacts with the barrier makes the procedure of following the peak of the packet not meaningful. Equation (32) is correct for a wave packet characterized by a narrow momentum distribution.

## 2.3 Calculation of the Dwell time for Buttiker problem

By calculating dwell time using (10) in Buttiker case it can be shown that

$$\tau_d = \tau_y \quad (33)$$

Therefore, the extent to which a spin precesses around the  $z$  axis is determined by the average dwell time of the particle in the barrier.

## 2.4 Traversal time of Buttiker

In the limit that the kinetic energy tends to zero,  $\tau_y$  vanishes and the traversal time

$$\tau_T = \tau_x = (\tau_y^2 + \tau_z^2)^{1/2} \quad (34)$$

is determined by  $\tau_z$ . We find from (28) for  $k = 0$

$$\tau_T = \tau_z = \frac{1}{2} \frac{\hbar}{V_0} \left( 1 + \frac{k_0 a}{\tanh(k_0 a)} \right) \quad (35)$$

For wide barrier

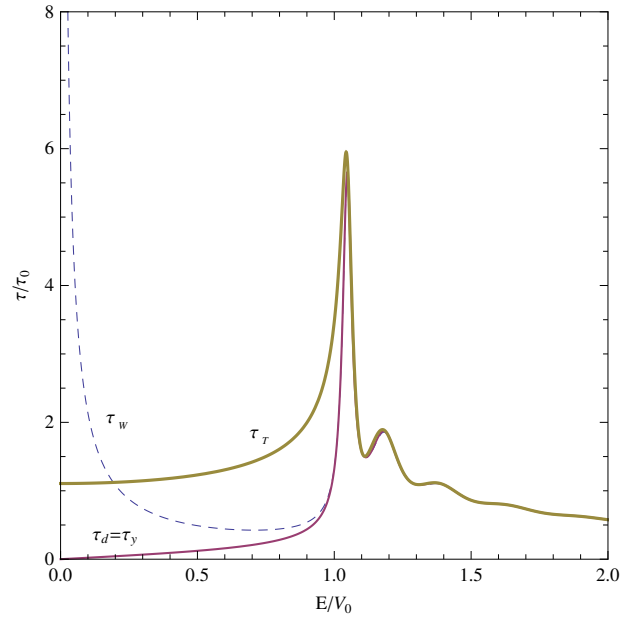
$$\tau_T = \frac{ma}{\hbar k_0} \quad (36)$$

For very thin barrier

$$\tau_T = \frac{\hbar}{V_0} \quad (37)$$

## 2.5 Comparison between times

Figure below compare between traversal time  $\tau_T$ , dwell time  $\tau_d = \tau_y$  and  $\tau_w$ . Whereas  $\tau_d = \tau_y$  tends to zero as  $k$  tends to zero,  $\tau_w$  diverges.



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