Sonic Black Hole In Bose-Einstein Condensate

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[1] Introduction

A sonic black hole is an analog to astrophysical black hole. Astrophysical black hole is a massive object surrounded by event horizon which light cannot escape from. A sonic black hole is a system with event horizon which phonons cannot escape from. A sonic black hole contains a region of subsonic flow ($v < c$) as well as a region of supersonic flow ($v > c$). Since phonons cannot propagate against the supersonic flow, the boundary between the two regions is the sonic black hole horizon.

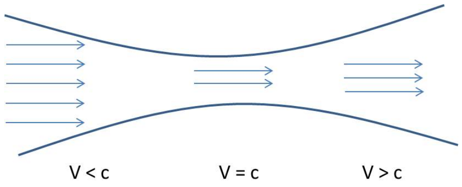

Figure 1: Hydrodynamic de Laval nozzle. For gas flowing from the left at subsonic velocity ($v < c$) the cross section becoming narrow until the waist. The gas accelerates as the cross section becoming narrow. If the gas will reach sonic velocity at the waist ($v = c$) then the outgoing flow will be supersonic ($v > c$).

[2] Classical equations of motion of a flow

The two equations that describe one dimensional compressible fluid that is subjected to external force are the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{1}$$

which is a mass conservation equation and Euler's equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{1}{m}\frac{\partial V(x)}{\partial x} \tag{2}$$

which is Newton's second law. Where ρ is the density, P is the pressure and v is the velocity. For a stationary flow, by using the divergence theorem, the continuity equation becomes

$$\frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx} + \frac{1}{\rho}\frac{d\rho}{dx} = 0 \tag{3}$$

where A is the cross section of the nozzle. Euler's equation (for a stationary flow) is

$$v\frac{dv}{dx} = -\frac{1}{\rho}\frac{dP}{dx} - \frac{1}{m}\frac{dV(x)}{dx} \tag{4}$$

If we assume that the flow is incompressible we can get Bernoulli's equation for incompressible fluid which is the energy conservation law

$$\frac{1}{2}v^2 + \frac{P}{\rho} + \frac{V(x)}{m} = const \tag{5}$$

——— [3] Speed of sound in a classical flow

We consider a flow without an external potential. We assume a standing flow with a small velocity v and small perturbations in the density and in the pressure:

$$\rho = \rho_0 + \delta\rho \tag{6}$$

$$P = P_0 + \delta P \tag{7}$$

Where ρ_0 and P_0 are constants and δ is a small quantity.
Under these assumptions and by neglecting second order terms (i.e. δ^2 and $v\delta$), equations (1) and (2) become

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial v}{\partial x} = 0 \tag{8}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{9}$$

We derive equation (8) with respect to t and neglect terms of second order:

$$\frac{\partial^2\rho}{\partial t^2} + \rho\frac{\partial^2 v}{\partial t\partial x} = 0 \tag{10}$$

We insert (9) into (10) and again neglect small terms:

$$\frac{\partial^2\rho}{\partial t^2} = -\rho\frac{\partial}{\partial x}\frac{\partial v}{\partial t} = \rho\frac{\partial}{\partial x}(\frac{1}{\rho}\frac{\partial P}{\partial x}) = \frac{\partial}{\partial x}\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial P}{\partial \rho}\frac{\partial^2\rho}{\partial x^2} \tag{11}$$

We define [1]:

$$c^2(x) \equiv \frac{\partial P}{\partial \rho} \tag{12}$$

We have got

$$\frac{\partial^2\rho}{\partial t^2} = c^2\frac{\partial^2\rho}{\partial x^2} \tag{13}$$

This is a wave equation for the variable ρ with the speed c, which is called the speed of sound.

——— [4] Hydrodynamic de Laval nozzle

We can now write equation (4) as

$$v\frac{dv}{dx} + c^2\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{m}\frac{dV(x)}{dx} = 0 \tag{14}$$

We now combine (3) and (14) for two different cases. First case is $dV(x)/dx = 0$ [1]:

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{1-(v/c)^2}\frac{1}{A}\frac{dA}{dx} \tag{15}$$

This case is illustrated in Fig.1. We can see that for a subsonic flow, the velocity increases as the cross section decreases and the velocity decreases as the cross section increases and opposite for supersonic flow. To archive a transition between subsonic flow to supersonic flow, the flow has to reach sonic flow exactly at the waist.
The second case is a constant cross section A. We get [2, 3]:

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{1-(v/c)^2}\frac{1}{mc^2}\frac{dV(x)}{dx} \tag{16}$$

Here, for subsonic flow, the velocity increases when the potential increases, and the velocity decreases as the potential decreases.

We can consider the following potential in the interval $x = [-L/2, L/2]$ [4]:

$$V(x) = V_0 \cos^2(\frac{2\pi x}{L}) \tag{17}$$

Under this potential, for a subsonic flow that flowing to the right, there is a black hole horizon at $x = 0$.

We now go back to equation (14) and write it as

$$\frac{d}{dx}\frac{1}{2}v^2 + c^2\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{m}\frac{dV(x)}{dx} = 0 \tag{18}$$

Integrating both sides with respect to x we get

$$\frac{1}{2}mv^2 + m\int \frac{c^2(x)}{\rho}\frac{d\rho}{dx}dx + V(x) = const \tag{19}$$

This is Bernoulli's equation for a compressible fluid.

======= [5] Quantum de Laval nozzle in the hydrodynamic approximation

The Gross-Pitaevskii equation for a Bose gas in a stationary state is

$$\mu\psi = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + g|\psi(x,t)|^2)\psi(x,t) \tag{20}$$

We write the wave function as a macroscopic order parameter:

$$\psi = \sqrt{n}e^{i\varphi} \tag{21}$$

Where $n(x)$ is the density. The probability current density is defined by

$$j = \frac{\hbar}{2mi}(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}) \tag{22}$$

giving

$$j = \frac{\hbar}{m} n \frac{\partial \varphi}{\partial x} \tag{23}$$

The velocity is defined by

$$v = \frac{j}{n} \tag{24}$$

giving

$$v = \frac{\hbar}{m} \frac{\partial \varphi}{\partial x} \tag{25}$$

The continuity equation for a stationary state is

$$\frac{\partial j}{\partial x} = 0 \tag{26}$$

Using (24) we can write it as

$$\frac{\partial}{\partial x}(nv) = 0 \tag{27}$$

Combining (20), (25) and (27) we get

$$\mu = -\frac{\hbar^2}{2m\sqrt{n}} \frac{\partial^2 \sqrt{n}}{\partial x^2} + V(x) + gn + \frac{1}{2} m v^2 \tag{28}$$

In the hydrodynamic approximation we assume that the interactions are strong, so the density varies slowly in x and we can neglect the term $\frac{\partial^2 \sqrt{n}}{\partial x^2}$. After doing so, (28) becomes

$$\frac{1}{2} m v^2 + gn + V(x) = \mu \tag{29}$$

This equation is an energy conservation law which is similar to Bernoulli's equation (19) with the identifications

$$c^2(x) \equiv \frac{gn(x)}{m}, \qquad \rho(x) \equiv mn(x), \qquad \mu = const \tag{30}$$

Under these identifications we can solve the integral in equation (19):

$$m \int \frac{c^2(x)}{\rho} \frac{d\rho}{dx} dx = m \int \frac{gn/m}{nm} \frac{d(nm)}{dx} dx = \int g \frac{dn}{dx} dx = gn \tag{31}$$

And we get the same equation. Thus, equation (29) can be derived to be the same as equation (16) which defines the de Laval nozzle.

In the next section we show that indeed the identifications we made in (30) are true.

[6] Small amplitude oscillations in a Bose gas at zero temperature

A one dimensional Bose-Einstein gas at zero temperature and external potential $V(x)$ is described by the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + g|\Psi(x,t)|^2)\Psi(x,t) \tag{32}$$

Where g is the effective one dimensional interaction parameter which is proportional to the interaction length.
We will now assume no external potential and take Ψ to be

$$\Psi(x,t) = \Psi'(x,t)e^{-\frac{i\mu t}{\hbar}} \tag{33}$$

Where μ is the chemical potential. For convenience we will not use the $'$ sign anymore. Equation (32) becomes

$$\mu\Psi + i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + g|\Psi|^2\Psi \tag{34}$$

Next step we take Ψ to be

$$\Psi(x,t) = \sqrt{n(x)} + \psi(x,t) \tag{35}$$

Where

$$|\psi| \ll \sqrt{n} \tag{36}$$

We insert (35) into (34) and neglect terms of second order in ψ to get

$$\mu\sqrt{n} + \mu\psi + i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\sqrt{n}}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x^2} + g\sqrt{n}n + 2gn\psi + gn\psi^* \tag{37}$$

But $\sqrt{n}$ is a solution to Gross-Pitaevskii equation

$$i\hbar\frac{\partial\sqrt{n}}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + gn)\sqrt{n} \tag{38}$$

So equation (37) becomes

$$\mu\psi + i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + 2gn\psi + gn\psi^* \tag{39}$$

We look for solutions of the form

$$\psi(x,t) = u(x)e^{-i\omega t} + v(x)^*e^{i\omega t} \tag{40}$$

By collecting all the terms evolving in time like $e^{-i\omega t}$ and $e^{i\omega t}$ one obtains the following pair of differential equations

$$\mu u(x) + \hbar\omega u(x) = -\frac{\hbar^2}{2m}\frac{\partial^2 u(x)}{\partial x^2} + 2gnu(x) + gnv(x) \tag{41}$$

$$\mu v(x) - \hbar\omega v(x) = -\frac{\hbar^2}{2m}\frac{\partial^2 v(x)}{\partial x^2} + 2gnv(x) + gnu(x) \tag{42}$$

We take the chemical potential to be the interaction energy

$$\mu = gn \tag{43}$$

and $u(x)$, $v(x)$ to be

$$u(x) = ue^{ikx}, \quad v(x) = ve^{ikx} \tag{44}$$

and we rearrange equations (41) and (42)

$$(\frac{\hbar^2 k^2}{2m} - \hbar\omega + gn)u + gnv = 0 \tag{45}$$

$$(\frac{\hbar^2 k^2}{2m} + \hbar\omega + gn)v + gnu = 0 \tag{46}$$

The determinant of the coefficients must vanish

$$(\frac{\hbar^2 k^2}{2m} + gn - \hbar\omega)(\frac{\hbar^2 k^2}{2m} + gn + \hbar\omega) - (gn)^2 = 0 \tag{47}$$

We get the dispersion relation [5]:

$$(\hbar\omega)^2 = (\frac{\hbar^2 k^2}{2m})^2 + \hbar^2 k^2 \frac{gn}{m} \tag{48}$$

For small wave numbers we neglect k^4 and get

$$\omega = k\sqrt{\frac{gn}{m}} \tag{49}$$

Thus, the group velocity is

$$v_g \equiv \frac{\partial \omega}{\partial k} = \sqrt{\frac{gn}{m}} \equiv c \tag{50}$$

And we proved the identifications we made in (30).

═══════ [7] Transonic solutions for the hydrodynamic approximation

We now wish to solve equation (29) and to find a transonic solution. For a stationary state, (24) and (26) gives

$$j = nv = const \tag{51}$$

We write (29) in terms of j

$$v^3 + v\frac{2(V(x) - \mu)}{m} + \frac{2jg}{m} = 0 \tag{52}$$

We take V(x) to be as defined in (17). The solutions for this equation are determined by the parameters V_0, μ, g, j. The way we choose these parameters is to get three real solutions. A complex solution is unstable. One of the three solutions is negative hence not physical. The two physical solutions are [4]:

$$v_-(x) = \sqrt{\frac{8(\mu - V(x))}{3m}} \cos\left(\frac{\theta(x) + 4\pi}{3}\right) \quad \text{subsonic solution} \tag{53}$$

$$v_+(x) = \sqrt{\frac{8(\mu - V(x))}{3m}} \cos\left(\frac{\theta(x)}{3}\right) \quad \text{supersonic solution} \tag{54}$$

Where

$$\cos(\theta(x)) = -\frac{jg}{m}\left(\frac{3m}{2(\mu - V(x))}\right)^{3/2} \tag{55}$$

From the velocity we can find the speed of sound. To do so we consider the relation we found earlier:

$$c(x) = \sqrt{\frac{gn(x)}{m}} \tag{56}$$

Together with

$$n(x)v(x) = \text{const} \Rightarrow \quad v(x) = \tfrac{const}{n(x)} \tag{57}$$

We write

$$c(x) = \sqrt{\frac{ga}{mv(x)}} \tag{58}$$

Where a is constant.

As mentioned before, the sonic black hole horizon is at $x = 0$, which is the maximum of the potential. The condition for transonic flow is given by the chemical potential [4]:

$$\mu_{critical} = \frac{3}{2}(jg)^{3/2} m^{1/3} + V_0 \tag{59}$$

The two solutions of the velocity are presented in figure 2 [4], together with the potential (17).

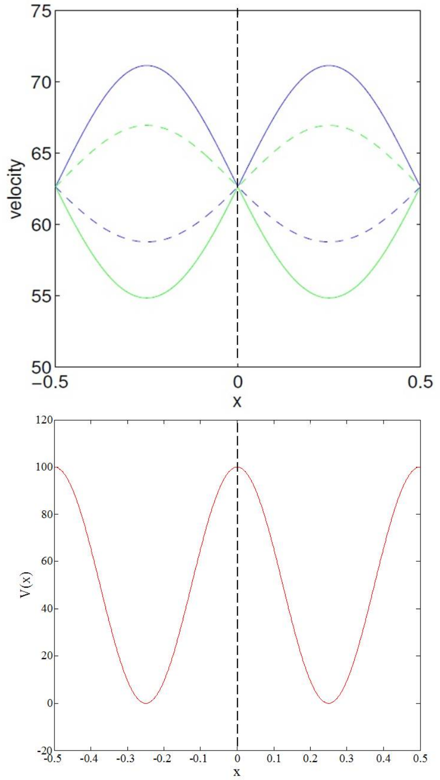

Figure 2: Upper panel is the transonic solution for (52). Green solid line is the subsonic branch, dashed green line is the corresponding speed of sound. Solid blue line is the supersonic branch with the dashed blue line describes the speed of sound. Lower panel is the potential V(x). The vertical dashed line represents the sonic black hole horizon.

References

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