

Self Organized Criticality

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==== [1] Introduction

A phase transition between gas and liquid is an example of a critical state, in order to sustain it we have to precisely tune the pressure, temperature, etc. Self Organized Criticality or SOC is a field of interest dealing with systems which drive themselves naturally into a state of criticality without the need of precise tuning of parameters as long as the spatial and time scales are big enough comparing to the driving force spatial and time scales.

The meaning of criticality is the fact that once the system reaches its critical state the correlation length diverges resulting in a correlation function (in our case for toppling sites) in the form of a power law. SOC presents in models from a large variety of fields such as geophysics, physical cosmology, evolutionary biology, economics, quantum gravity, etc.

==== [2] Sand pile model

SOC was first developed by Bak, Tang and Wiesenfeld in their famous paper from 1987 "SOC: an explanation of 1/f noise". Bak et al. proposed a simple example of a driven system whose natural dynamics drives it towards, and then maintains it, at the edge of stability: a sand pile. It is observed that for dry sand, one can characterize its macroscopic behavior in terms of an angle θ_c called the angle of repose, which depends on the detailed structure (distribution sizes, shapes and roughness etc.) of the constituting grains. If we make a sand pile in which the local slope is smaller than θ_c everywhere, such a pile is stable. On such a pile, addition of a small amount of sand will cause only a weak response, but adding a small amount of sand to a configuration where the average slope is larger than θ_c , will often result in an avalanche whose size is of the order of the system size. In a pile where the average slope is θ_c , the response to addition of sand is less predictable. It might cause almost no relaxation, or it may cause avalanches of intermediate sizes, or a catastrophic avalanche which affects the entire system. Such a state is critical. BTW observed that if one builds a sand pile by pouring sand very slowly, say on a flat circular table one gets a conical shaped pile, with slope equal to θ_c . The system is invariably driven towards its critical state: it shows SOC.

Let us now introduce a mathematical model for the sand pile. We introduce a $d = 2$ lattice in the size of $S \times L$ with periodic boundary conditions in the S direction. each site x has an integer Z . we define the coordinate of a site $x = (x_1, x_2)$ as $T(x) = x_1 + x_2$ and consider a model of finite lattice confined to the region $0 \leq T < L$. the number of sites on each constant T surface being S . particles are added at random on the "top" (i.e. at the surface $T = 0$) and drop off the pile at $T = L$.

The iteration rules are as follow:

$$\text{dropping} \rightarrow \begin{array}{l} Z(x) = Z(x) + 1 \\ \text{for random } (x | T(x) = 0) \end{array}$$

$$\text{avalanche} \rightarrow \begin{array}{l} \text{if } Z \geq Z_c = 2 \\ \text{then toppling} \end{array} \quad \begin{array}{l} \text{recursive} \\ Z(x) = Z(x) - 2 \\ Z(x') = Z(x') + 1 \text{ for } x' = \begin{array}{l} x + (1, 0) \\ x + (0, 1) \end{array} \end{array}$$

fig1

When $Z(X)$ exceeds Z_c the column topples and the effect of a toppling is to create an avalanche, the avalanche is characterized by:

$t = \text{duration of avalanche}$

$m = \text{mass of avalanche}$

The duration t , is the number of constant T surfaces affected, while the mass m , is the total number of sites that toppled. Our question is therefore: what is the distribution of duration t and of mass m ? In order to answer this question we'll take a look at a typical avalanche:

fig2

It's crucial to notice two facts when examining a three sites triangle (two nearby ones in the same surface T and the one just beneath them in the surface $T + 1$): 1. because of the dimension ($z_c = d = 2$) there is no way of having the two nearby ones topple without the third one being topple as well. 2. there is a probability p for the $T + 1$ surface one to topple if only one of the T surface ones topples, and a probability of $1 - p$ for the $T + 1$ surface one to not topple if only one of the T surface ones topples.

These observations leads us to the conclusion that every cluster of toppled sites in an avalanche has no holes and it's area equals it's mass m . The last T surface of the avalanche always contains only a single toppling. furthermore, the boundary of such a cluster is formed by the path of two annihilating random walkers starting in the origin, each taking steps along \vec{e}_1 and \vec{e}_2 with equal probability.

Instead of dealing with two random walkers, we can consider only the single random walk performed by their relative positions, or the difference walk. This walk is characterized by the transition probabilities:

$$P(l \rightarrow l + 1) = p_+ = p^2$$

$$P(l \rightarrow l) = p_0 = 2p(1 - p)$$

$$P(l \rightarrow l - 1) = p_- = (1 - p)^2$$

Where l is the distance between the walkers. At the beginning $l = 1$ and at the end $l = 0$, we'll first calculate the probability g_t to achieve this state (i.e. the termination of the avalanche) after exactly t steps:

$$g_t = \sum_{\sigma=0}^{t-1} C_{t-1}^{\sigma} p_0^{\sigma} \tilde{g}_{t-\sigma}$$

$$\tilde{g}_k = \frac{1}{k} p_+^{\frac{k-1}{2}} p_-^{\frac{k+1}{2}} C_k^{\frac{k+1}{2}}$$

We now define the quantity $p_+ - p_- = -2\epsilon$. Under the assumption that for $t \gg 1$ we get $p_+ - p_- = -2\epsilon \ll 1$, we can replace the sum by an integral and get that:

$$g_t \propto \frac{1}{t^{\frac{3}{2}}} e^{-4t\epsilon^2} \equiv \frac{1}{t^{d-2+\eta}} e^{-\frac{t}{\xi}} \text{ whereas } \xi \propto |\Delta|^{-\nu}$$

We can now identify g_t as a correlation function! Correlating the toppling of two sites: the single on of the surface $T = 0$ with the single one of the surface $T = t - 1$, Whereas $d = 2$ is the dimension, $\eta = \frac{3}{2}$ is a critical exponent, $\xi = (2\epsilon)^{-2}$ is the correlation length and $\nu = 2$ is the critical exponent associated with the correlation length. Since $\epsilon_{t \rightarrow \infty} \rightarrow 0$ and therefore $\xi_{t \rightarrow \infty} \rightarrow \infty$ it's now evident that the system presents SOC! In the SOC state g_t takes the form:

$$g_t \propto t^{-\frac{3}{2}} = \frac{1}{t^\eta}$$

$P_w(t)$ is the sum of all graphs that have not terminated after t steps which is the distribution of duration.

$$P_w(t) = 1 - \int_0^t g_t dt \propto t^{-\eta+1} \equiv t^{-\alpha} = t^{-\frac{1}{2}}$$

Now after finding the distribution of duration $P_w(t)$ it's easy to find the distribution of mass $P_w(m)$. let $\tilde{m}(t)$ be the number of sites that topple on the surface $T = t$ given that at least one site does, It's clear that each grain that enters the system must leave, accordingly the flux of each T surface must stay constant therefore:

$$\tilde{m}(t)P_w(t) = \tilde{m}(t)t^{-\alpha} = \text{const} \rightarrow \tilde{m}(t) \propto t^\alpha$$

The total mass m is given by:

$$m = \int \tilde{m}(t) dt \propto t^{\alpha+1} \rightarrow t \propto m^{\frac{1}{\alpha+1}}$$

Changing variables leads us to the distribution of mass:

$$P_w(t) \propto t^{-\alpha} \rightarrow P_w(m) \propto m^{-\frac{\alpha}{1+\alpha}} \equiv m^{-\beta} = m^{-\frac{1}{3}}$$
