Levy Flights and Continuous Time Random Walks

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Notation remark
$f(x)$ is a probability distribution function (PDF) of the random variable x.
$f(t)$ is a PDF of the random variable t.
$f(x,t)$ is a common PDF of x and t.
$\tilde{f}(k)$ is a Fourier transform of $f(x)$.
$\tilde{\phi}(s)$ is a Laplace transform of $f(t)$.

Levy flight is a random walk in which the step lenghts are distributed according to a Levy distribution (defined later).
Continuous Time Random Walks (CTRW) is a random walk in which the step lenghts and the waiting times are distributed according to a Levy distributions.

Problem Definishion

A. Derive an exprassion for $f_N(x)$,the probability for position x after N steps, and $x(N)$ for a Levy flights motion.
B. Derive an expression for $f(x,t)$,the probability for position x at time t, and $x(t)$ for an Continuous Time Random Walks motion.
Levy stable distribution, $f(x;a,\alpha)$, is defined by its characteristic function

$$\tilde{f}(k;a,\alpha) =< \exp(ikx) >= \exp(-|ak|^{\alpha}) \tag{1}$$

Levy distribution is defined for $0 < \alpha < 2$, otherewise $f(x;a,\alpha)$ is not normalized ($\alpha < 0$) or gets negative values and cannot be a pdf ($Var(f(x;a,\alpha)) = 0$). here a have the units of length.
For $f(t;a,\alpha)$ (a have the units of time) which is defined through its Laplace transform, $\tilde{\phi}(s;a,\alpha)$, $f(t;a,\alpha)$ is non negative for $0 < \alpha < 1$.
For $\alpha = 1$ and $\alpha = 2$ we get two well known special cases, Lorentzian and Gaussian

$$f(x;a,1) = \frac{a}{\pi(a^2+x^2)} \quad f(x;\frac{\sigma^2}{2},2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{x^2}{2\sigma^2}) \tag{2}$$

Width the width of the PDF $f(\frac{x}{N};a,\alpha)$ defined as N
Stability a random variable is stable if a linear combination of two independent copies of the variable has the same distribution. $f_N(\sum x_i;a,\alpha) = f(x_1;a,\alpha) * f(x_2;a,\alpha) ... * f(x_N;a,\alpha)$, $*$ stands for convolution,
in Fourier space $\tilde{f_N}(k) = \prod \tilde{f}(k_i;a,\alpha)$, for Levy stable distribution one gets
$\tilde{f_N}(k) = \exp(-|aN^{\frac{1}{\alpha}}k|^{\alpha}) \rightarrow f_N(\sum x_i;a,\alpha) = f(\frac{x}{N^{\frac{1}{\alpha}}};a,\alpha)$, that is the generalized center limit theorem.

in context of the Levy flight the width is the flight length

$$x(N) \sim N^{\frac{1}{\alpha}} \tag{3}$$

An important equation that we will use is the Montroll-Weiss equation

$$\tilde{f}(k,s) = \frac{1-\tilde{f}(s)}{s}\frac{1}{1-\tilde{\phi}(s)\tilde{f}(k)} \tag{4}$$

* Montroll-Weiss Equation derivation is in the appendix at the end of the lecture.

For anomalous diffusion we assume the waiting times PDF $\tilde{\phi}(s;\tau,\beta)$, and step length PDF $\tilde{f}(k;\lambda,\alpha)$. Expand the PDFs for small k and s and put it into the Montroll-Weiss equation and taking the first order terms gives

$$\tilde{f}(k,s) \sim \frac{\tau s^{\beta-1}}{\tau s^{\beta} + \lambda k^{\alpha}} \tag{5}$$

As an example one can verify the well known normal diffusion kernal, i.e $\alpha = 2$,$\beta = 1$, $\lambda = \frac{\sigma^2}{2}$

$$f(x,t) = \frac{1}{\sqrt{\frac{2\pi\sigma^2 t}{\tau}}}\exp(-\frac{\tau x^2}{2\sigma^2 t}) \;\rightarrow\; \tilde{f}(k,s) \sim \frac{1}{s+\frac{\sigma^2}{2\tau}k^2} \tag{6}$$

where $\frac{\sigma^2}{2\tau}$ is the diffusion constant.
transforming back to the space-time gives

$$f(x,t) \sim \int_0^{\infty}\int_{-\infty}^{\infty} dkdse^{-ikx-st}\frac{\tau s^{\beta-1}}{\tau s^{\beta}+\lambda k^{\alpha}} = \int_{-\infty}^{\infty} dke^{-ikx}E_{\beta}(-k^{\alpha}t^{\beta}) \tag{7}$$

$E_{\beta}(-k^{\alpha}t^{\beta})$ (Mittag-Leffler function) defined by

$$E_{\beta}(z) = \sum \frac{z^n}{\Gamma(1+\beta n)} \tag{8}$$

One can see the Fourier function as $\tilde{f}(t^{\beta}k^{\alpha})$, i.e the space coordinate function is $f(\frac{x^{\alpha}}{t^{\beta}})$.
From here one can extract the width time dependance

$$x(t) \sim t^{\frac{\beta}{\alpha}} \tag{9}$$

according to the power low of $x(t)$ we can classify the different diffusion types.
$2\beta = \alpha$ is the normal diffusion case, $2\beta > \alpha$ is the superdiffusion case, $2\beta < \alpha$ is the superdiffusion case.

Appendix

Montroll-Weiss Equation

The equation is in the context of a random walker.
In order to derive the equation we will define few PDFs

1. $P(t) = 1 - \int_0^t d\tilde{t} f(\tilde{t})$ - the probability that no step is taken during a time t
2. $Q(x,t)$ - the probability that the walker arrived at position x at time t
3. $f(x,t) = \int_0^t d\tilde{t} P(t-\tilde{t})Q(x,\tilde{t})$ - the PDF that the walker at position x at time t.

$Q(x,t)$ can be written as follow: $Q(x,t) = f(x - \tilde{x)} f(t-\tilde{t})Q(\tilde{x},\tilde{t})$ for a constant $(\tilde{x},\tilde{t})$ - the last step length and time.
However we have to integrate over all posible last steps, i.e

$$Q(x,t) = \int_{-\infty}^{x} d\tilde{x} \int_0^t d\tilde{t} f(x-\tilde{x})f(t-\tilde{t})Q(\tilde{x},\tilde{t}) + \delta(x)\delta(t) \tag{10}$$

where the last term takes into acount the initial condition $Q(x=0,t=0)=1$, $f(\tilde{t})$=0 for $t \leq \tilde{t}$.
note that $Q(x,t)$ is a convolution of $f(x), f(t)$ and $Q(x,t)$ with respect to both, x and t
while $f(x,t)$ is a convolution of F and Q with respect to t only.

Using the convolution theorem, we can fined Fourier-Laplace transform of $Q(x,t)$ and $f(x,t)$

$$\tilde{Q}(k,s) = \tilde{f}(k)\tilde{\phi}(s)\tilde{Q}(k,s) + 1 \to \tilde{Q}(k,s) = \frac{1}{1-\tilde{f}(k)\tilde{\phi}(s)} \tag{11}$$

$$\tilde{f}(k,s) = \tilde{P}(s)\tilde{Q}(k,s) \tag{12}$$

$$\frac{dP(t)}{dt} = \delta(t) - f(t) \ \to \ s\tilde{P}(s) = 1 - \tilde{\phi}(s) \tag{13}$$

$$\tilde{P}(s) = \frac{1-\tilde{\phi}(s)}{s} \tag{14}$$

assign $\tilde{Q}(k,s)$ and $\tilde{P}(s)$ into $\tilde{f}(k,s)$ gives

$$\tilde{f}(k,s) = \frac{1-\tilde{\phi}(s)}{s}\frac{1}{1-\tilde{f}(k)\tilde{\phi}(s)} \tag{15}$$

References

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