# Anderson localization

## Bloch's theorem

consider a Hamiltonian of the following form:

$$\hat{H} = \begin{bmatrix} \epsilon_1 & 1 & 0 & \cdots & 0 \\ 1 & \epsilon_2 & 1 & \cdots & 0 \\ 0 & 1 & \epsilon_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & \epsilon_N \end{bmatrix} \tag{1}$$

First we consider a case of a periodic lattice where $\epsilon_n = \epsilon$ for all sites. We can now write $N$ equations for the eigenfunctions:

$$c_{n+1} + c_{n-1} + \epsilon c_n = E c_n \tag{2}$$

weher $c_n = \langle n|\psi\rangle$ is the value of the corresponding function at site $n$.

In periodic lattice, Bloch's theorem is applied and we get the amplitudes

$$c_{n+R} = c_n e^{ikR} \tag{3}$$

subtituting them in the eigenfunction equations we obtain:

$$E = \epsilon + 2\cos(k) \tag{4}$$

The DOS (density Of states) is defined as the reciprocal velocity:

$$\frac{1}{v_E} = \frac{dk}{dE} = -\frac{1}{2\sin(k)}$$

substituting $\sin(k)$ as $\sqrt{1 - \frac{E^2}{4}}$ (Eq.4 when $\epsilon = 0$) we obtain:

$$\frac{1}{v_E} = \frac{1}{\sqrt{4 - E^2}}$$

## impurities

Consider the same hamlitonain as in bloch's case with one impurity at the site $n = 0$ with potential energy $\epsilon_0$ and $\epsilon = 0$ for the rest of the sites:

$$\hat{H} = \begin{bmatrix} \epsilon_0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

If we diagonalize this hamiltonian we obtain a specific enrgy level $E = (\epsilon_0^2 + 4)^{1/2}$ and the rest of the enrgies satisfy $|E| < 2$. According to Eq.2 and Eq.4 all the energies that satisfy

$|E| < 2$ belong to Bloch's states and have periodic solutions for $c_n$'s. We will now check what happen in the case of $E = (\epsilon_0^2 + 4)^{1/2}$ which has to be related to the site $n = 0$. In order to find the probability amplitudes $c_n$'s in this case we will substitute the following expression:

$$c_n = \begin{cases} Ae^{-\gamma n} & n > 0 \\ Be^{\gamma n} & n < 0 \end{cases}$$

Substituting this into Eq. 2 with $n = 0$ and $\epsilon_n = \epsilon_0$ gives

$$E = 2e^{\gamma} + \epsilon_0$$

which means we get a real positive $\gamma$. The decreasing probability amplitude implies localization in the region of $n = 0$

If the system is localized with one impurity then we assume it is localized with more impurities. specifically, we obtain an expression for the amplitudes if the system is localized in the region of $n_0$:

$$c_n = c_{n_0} e^{-\frac{n-n_0}{l}}$$

where $l$ is the localization length.

# Transfer Matrix Method

The $N$ simultaneous equations above can be written in a matrix form:

$$\begin{pmatrix} c_{n+1} \\ c_n \end{pmatrix} = T_n \begin{pmatrix} c_n \\ c_{n-1} \end{pmatrix} \tag{5}$$

where

$$T_n = \begin{pmatrix} E - \epsilon_n & -1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

by multiplying the matrix $n$ times, we can find the $n$'th amplitude from the initial amplitudes

$$\begin{pmatrix} c_{n+1} \\ c_n \end{pmatrix} = \prod_{k=1}^{n} T_k \begin{pmatrix} c_1 \\ c_0 \end{pmatrix} \tag{7}$$

for hard wall at site $n = 0$ boundary condition we get $c_0 =$ and $c_{N+1} = 0$. Solving the matrix above interatively for $c_N$ we obtain a polynomail expression:

$$c_N = A \prod_{n=0}^{N-1} (E - E_n) \tag{8}$$

this expression can also be written as:

$$c_N = A \prod_{n=0}^{N-1} |E - E_n| e^{i\pi\theta(E_n - E)} \tag{9}$$

where $\theta$ is the heaviside function. we will now define

$$L(E) = \lim_{N\to\infty} \frac{1}{|N|} \ln \left| \frac{c_N(E)}{c_0} \right| \tag{10}$$

and $L(E)$ becomes:

$$L(E) = \frac{1}{N}\sum_n \ln|E-E_n| + \frac{i\pi}{N}\sum_n \theta(E_n-E) \approx \int_{-\infty}^{+\infty} dE'\rho(E')\ln|E'-E| + i\pi\int_{-\infty}^{+\infty} dE'\rho(E')\theta(E'-E) \tag{11}$$

For a large system where $N \to \infty$, $\rho(E)$ is the averaged density of states. Using the identity of the heaviside function, we can now write

$$\int_{-\infty}^{+\infty} dE'\rho(E')\theta(E'-E) = \int_{E}^{+\infty} dE'\rho(E') \equiv I(E) \tag{12}$$

Eq. 11 can now be written

$$L(E) = \int_{-\infty}^{+\infty} dE'\rho(E')\ln|E'-E| + i\pi I(E) \tag{13}$$

Notice that in Eq. 12 $I(E)$ is the integrated DOS and we can find it from Eq.13 relation

$$I(E) = \mathrm{Im}\left[\frac{L(E)}{\pi}\right] \tag{14}$$

and therefore the DOS can be extraceted from it:

$$\rho(E) = -\frac{dI(E)}{dE} \tag{15}$$

Since the $c_n$'s are complex numbers they can be written in the form $c_n = |c_n|e^{i\phi_n}$. subtituting this into the equation of $L(E)$ (Eq. 10) we obtain:

$$L(E) = \lim_{N\to\infty} \frac{1}{N}\left[\ln\left|\frac{c_N}{c_0}\right| + i(\phi_n - \phi_0)\right] \tag{16}$$

We now notice that the reciprocal localization length,$l$ is the real part of $L(E)$ and therefore:

$$\frac{1}{l} = \mathrm{Re}[L(E)] = \int_{-\infty}^{+\infty} dE'\rho(E')\ln|E-E'| \tag{17}$$

where $\rho(E)$ is the everaged DOS $\langle\rho(E)\rangle$

# green function method

Let $\mid \Psi^\alpha\rangle$ be the eigenstates of the hamiltonian and $\mid \Psi_n\rangle$ the eigenstates of the unperturbed hamiltonian, $\hat{H}_0$, when both $\alpha$ and $n$ go from 1 to $N$ and $\hat{H} = \hat{H}_0 + \hat{V}$. We can now write the resolvent in the basis of $\mid \Psi_n\rangle$ :

$$G_{nm} = \langle\Psi_n \mid \hat{G} \mid \Psi_m\rangle = \sum_\alpha \langle\Psi_n \mid \Psi^\alpha\rangle\langle\Psi^\alpha \mid \hat{G} \mid \Psi^\alpha\rangle\langle\Psi^\alpha \mid \Psi_m\rangle \tag{18}$$

The overlap $\langle\Psi_n \mid \Psi^\alpha\rangle$ is the probability amplitude for state $n$ with total enrgy of the system $E^\alpha$ and therefore equals to $c_n(E^\alpha)$. Replacing $\hat{G}$ with $\frac{1}{E\hat{I}-\hat{H}}$ we get:

$$G_{nm} = \sum_{\alpha} \frac{c_n(E^\alpha)c_m(E^\alpha)}{E - E^\alpha} \tag{19}$$

specifically, we have a particular interest in finding $G_{1N}$ for later use:

$$G_{1N} = \sum_{\alpha} \frac{c_1(E^\alpha)c_N(E^\alpha)}{E - E^\alpha} \tag{20}$$

we will also find $G_{1N}$ by diret calculation of the matrix:

$$G_{1N} = \frac{C_{1N}}{det|EI - H|} \tag{21}$$

where $C_{1N}$ is the cofactor of $EI - H$ of the $1, N$ element. The calculation is easy if it is made directly from the hamiltonian, and we get:

$$G_{1N} = 1/\prod_{\alpha=1}^{N}(E - E^\alpha) \tag{22}$$

if we equate terms we get:

$$\sum_{\alpha} \frac{c_1(E^\alpha)c_N(E^\alpha)}{E - E^\alpha} = 1/\prod_{\alpha=1}^{N}(E - E^\alpha) \tag{23}$$

we will now make a contour integration around the pole $E = E^\beta$. All the terms in the left side of the equation will cancel since they are not inside the contour excpet for the pole $E^\alpha = E^\beta$. And their residue is $c_1(E^\beta)c_N(E^\beta)$. The residue on the right side is also of one pole and it's equal to $1/\prod_{\alpha\neq\beta}(E - E^\alpha)$. So we have

$$c_1(E^\beta)c_N(E^\beta) = 1/\prod_{\alpha\neq\beta}(E^\beta - E^\alpha) \tag{24}$$

If the localization is in the region of the $n_0$'th state, we can conclude that $c_1(E^\beta) = c_{n_0}(E^\beta)e^{-\frac{n_0-1}{l(E^\beta)}}$ and $c_N(E^\beta) = c_{n_0}(E^\beta)e^{-\frac{N-n_0}{l(E^\beta)}}$ and we get:

$$\left(c_{n_0}(E^\beta)\right)^2 e^{-\frac{N-1}{l(E^\beta)}} = 1/\prod_{\alpha\neq\beta}(E^\beta - E^\alpha) \tag{25}$$

and:

$$2\ln(c_{n_0}(E^\beta)) - \frac{N-1}{l(E^\beta)} = -\sum_{\alpha\neq\beta}\ln(E^\beta - E^\alpha) \tag{26}$$

Taking $N \to \infty$ the left hand side term is cancelled and we get:

$$\frac{1}{l} = \lim_{N\to\infty}\frac{1}{N-1}\sum_{\alpha\neq\beta}\ln(E^\beta - E^\alpha) \tag{27}$$

When $N$ goes to infinty we can replace the summation with integral, adding the density of states:

$$\frac{1}{l(E^\beta)} = \int \rho(E^\alpha) \ln(E^\beta - E^\alpha) dE^\alpha \tag{28}$$

and the general solution for the localization length can noe be rewritten:

$$\frac{1}{l(E)} = \int \rho(E') \ln(E - E') dE' \tag{29}$$

# Localization length for weak disorder

For a weak disorder we can write again the equation for tight binding hamiltonian

$$c_{n+1} + c_{n-1} + \lambda \epsilon_n c_n = E c_n \tag{30}$$

where $\lambda << 1$
We will now define $R_n$:

$$R_n \equiv \frac{c_n}{c_{n-1}} \tag{31}$$

subtituting it in Eq. 30 we obtain:

$$R_{n+1} = (E - \lambda \epsilon_n) - \frac{1}{R_n} \tag{32}$$

The definition of the localization length with $R_n$ is now:

$$\frac{1}{l} = \lim_{N \to \infty} \frac{1}{N} \langle \sum_{n=1}^{N} \ln |R_n| \rangle \tag{33}$$

To find the localization length, we will noe expand $R_n$ in the following way:

$$R_n = A e^{\lambda B_n + \lambda^2 C_n + \lambda^3 D_n + ...} \tag{34}$$

where $A$ is unknown.
Subtituting it in Eq.32 we get three leading orders of $\lambda$:

$$\lambda^0 : A = E - A^{-1} \tag{35}$$

$$\lambda^1 : A B_{n+1} = \epsilon_n - A^{-1} B_n \tag{36}$$

$$\lambda^2 : A \left[ C_{n+1} + \frac{1}{2} B_{n+1}^2 \right] = A^{-1} \left[ C_n - \frac{1}{2} B_n^2 \right] \tag{37}$$

for $A$ we get:

$$A = \frac{1}{2} \left[ E + -\sqrt{E^2 - 4} \right] \tag{38}$$

to find the rest of the parameters, for a random distributed energies per site we define $\langle \epsilon_n \rangle = 0$ and $\langle \epsilon_n^2 \rangle = \sigma^2$ without loss of generality:

$$\langle B_n = 0 \rangle = 0 \tag{39}$$

$$\langle B_n^2 \rangle = \frac{A^2}{A^4 - 1}\sigma^2 \tag{40}$$

$$\langle C_n \rangle = -\frac{1}{2}\frac{A^2}{(A^2-1)^2}\sigma^2 \tag{41}$$

substituting the three leading orderes in Eq.33 we obtain:

$$\frac{1}{l} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[\langle \ln A\rangle + \lambda\langle B_n\rangle + \lambda^2\langle C_n\rangle + ...\right] \approx \frac{1}{2}\frac{\sigma^2}{4-E^2} \tag{42}$$

If the enrgies rae defined in an interval $\epsilon_n \ \left[-\frac{W}{2}, \frac{W}{2}\right]$ then the variance is

$$\sigma^2 = \frac{W}{12} \tag{43}$$

Substituting it, we get:

$$l = 24\frac{4-E^2}{W^2} \tag{44}$$

## Finding the localization length with the resolvent

We will find the localization length again using the complex resolvent method as was derived by Thouless. The real part of the resolvent is:

$$g(E) = \frac{1}{N}\Re[tr(G^+)] = \sum_k \frac{1}{E - E_k} \tag{45}$$

and he imaginary part:

$$\rho(E) = \frac{1}{N}\Im[tr(G^+)] = \sum_k \delta(E - E_k) = \frac{1}{v_E} \tag{46}$$

The zero order resolvent is defined as follow:

$$G^0_{nn} = \frac{1}{N}\sum_k \langle k|\frac{|k\rangle\langle k|}{E - E_k + i0}|k\rangle = \frac{1}{N}\sum_k \frac{1}{E - E_k + i0} \tag{47}$$

The real part is summed to zero and the imaginary part $\rho(E)$ is exactly the DOS, $\rho(E) = \frac{1}{v_E}$. and therefore we get:

$$G^0_{nn} = \frac{i}{v_E} \tag{48}$$

Eq. 27 for finding the localization length can be written as follow:

$$\frac{1}{l} = \frac{1}{N}\sum_k \ln(E - E_k) = \int^E g(E')dE' \tag{49}$$

we will now study the localization length for a small region of energy compared to the hopping amplitude ($W \ll 1$). In this cas we can expand the resolvent to second order:

$$G_{nn} = \sum_n G^0_{nn} + \sum_n \sum_m G^0_{nm}\epsilon_m G^0_{mn} + \sum_n \sum_m \sum_l G^0_{nm}\epsilon_m G^0_{ml}\epsilon_l G^0_{ln} + \cdots \tag{50}$$

Since $G^0_{nn} = \frac{i}{v_E}$ is imaginary, we don't use the first term, because we are looking for the real part $g(E)$ to replace in Eq.49.0 The second term averages to zero. the third term, with the connection $< \epsilon_n\epsilon_m >= (W^2/12)\delta_{nm}$ reduces to:

$$(W^2/12)\sum_n \sum_m G^0_{nm}G^0_{mm}G^0_{mn} = (W^2/12)\sum_m G^0_{mm}((G^0)^2)_{mm} \tag{51}$$

From Eq.47 we can write $\frac{dG^0_{mm}}{dE} = ((G^0)^2)_{mm}$ and substituting it in Eq.51. and back into Eq.50. eventually we get:

$$\Re(G_{nn}) = -(W^2/12)\sum_m G^0_{mm}\frac{dG^0_{mm}}{dE} \tag{52}$$

The diagonal element of the non-perturbated resolvent can be written:

$$G^0_{nn} = \frac{i}{v_E} = \frac{i}{\sqrt{E^2-4}} \tag{53}$$

subtituting this term back into Eq.52 , we get a real solution which is $g(E)$. substituting this in Eq.49 we get:

$$\frac{1}{l(E)} = (W^2/24)(4-E^2)^{-1} \tag{54}$$

$W \ll 1$ implies $E \ll 1$ and therefore the last equation is

$$\frac{1}{l(E)} = W^2/96 \tag{55}$$

# Numerical results for three types of distribution

The numerical experiment has been conducted using a $3000 \times 3000$ Hamiltonian with three different types of distributions. The first is a box distribuition where the enrgy of each site $\epsilon_i$ has been chosen randomically from a $[-\frac{W}{2}, \frac{W}{2}]$ region. The second is a Normal distribution where the enrgies $\epsilon_i$ has been chosen randomically with a known $\sigma^2$. Since we didn't want ambiguities while forming the normal distribution, We used a normal distribution where $\sigma = 1$ and $\mu = 0$ and divided it into 3000 parts of equal area (by integration). the values ($x_i$'s) were scrambled and then for controlling the variance we used $\epsilon_i = \sigma x_i$
The third is a Log-Normal distribution with known $\sigma$ and $\sigma_s$, where $\sigma_s$ is the the variance of $\log \epsilon_i$ distribution.for finding its values we used the same method as before, taking the same $x_i$'s and using $\epsilon_i = e^{\mu+\sigma_s x_i}$ where $\sigma^2 = \left(e^{\sigma_s^2}-1\right)e^{2\mu+\sigma_s^2}$
We have used the method of participation number to calculate the localization of a given state. After calculating the eigenstates numerically, we calculated the participation number:

$$P_n = \frac{1}{\sum_n C_n^2}$$

where the $C_n^2$'s indicate the probability to find the particle in site $n$. The number we obtain describes the number of the most probable sites, i.e. the localization length.

The first figure is an example for the method of work. We have used the box distribution where Each cyan square indicates the participation number for a varing energy $E$ at the box distribution. The red squares indicates the average of the participation numbers of the different energies.

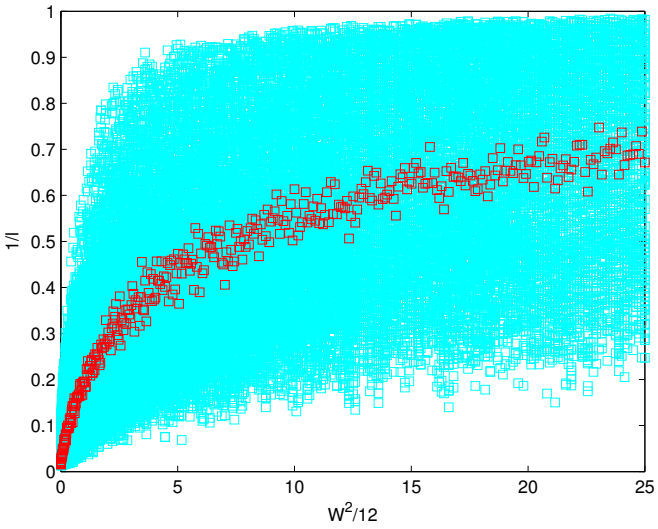


Figure 1: Box distribution with $W^2/12$ max $= 25$

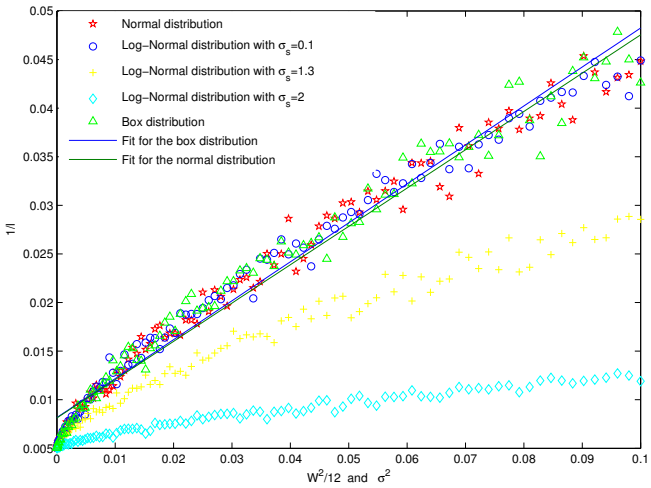


Figure 2: Box distribution with $W^2/12$ max $= 0.1$, Normal distribution and log-normal distribution with $\sigma^2_{max} = 0.1$

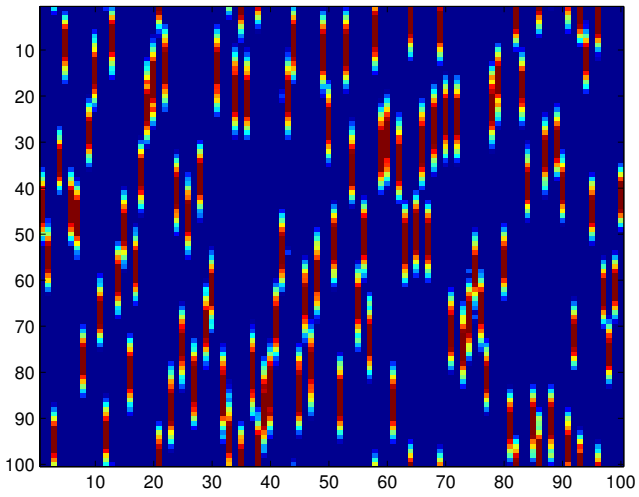


Figure 3: Box distribution with $W = 100$

## Results and discussion

fig.2 shows five different experiments. The green triangles are the average PP of the box distribution with varying $W$'s up to $\frac{W^2}{12} = 0.1$ and its linear fitting. The red polygones show the corresponding Normal distribution with $\sigma_{max} = 0.1$ and its linear fitting. The blue, yeloow and cyan marks indicate the log-normal distribution with corresponding $\sigma_s = 0.1$,1.3 and 2. The linear line that was fitted to calculate the relation between $W^2/12$ and $\frac{1}{l}$ and also between $\sigma^2$ and $\frac{1}{l}$ . since the method of participation number is not valid for small or large $W$'s, we did not calculate the ratio $\frac{1}{8}$ which was obtained in the pervious section. however, one can notice that a linear line is fitted well for both distributions. also, Since $W^2/12$ and $\sigma^2$ are both the RMS of each distribution, we see almost the same results for the linear fitting. another interesting result is the rate of the strength of loclization. We can see clearly The the localiztion become stronger for the box and normal distribution as we increase the variance. for the log-normal distribution, the localiztion stays weak. This is attributed to the fact that most of the energies are distributed in the same region and thefore the the disorder is very weak. The disorder is hardly affected from the energies at the edge of the band even though they are much larger than the energies in the middle of the band.
Figure 3 illustartes the localization with imaging of the eigenstates of a box distribution with $W = 100$ of a $100 \times 100$ Hamiltonain. The 100 differefnt eigenstate are aligned vertically by imaging where the red regions are the regions of localization.

## Quasi 1D disordered system

In the case of $\lambda > a$ where $\lambda$ is the wave length and $a$ is the correlation length, the tight binding model is no longer fit. the coupling between neighbouring sites exceeds the limit of only near neighbour site. We use a quasi 1D disordered Hamiltonian to solve this problem. The hamiltonain is similar to the previous one but there are more off diagonal elements. we

write it as follow(example with $b = 2$):

$$\hat{H} = \begin{bmatrix} \epsilon_1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & \epsilon_2 & 1 & 1 & \cdots & 0 \\ 1 & 1 & \epsilon_3 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \epsilon_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 1 & \epsilon_N \end{bmatrix} \tag{56}$$

where $b$ is the number of the propagating channels. in this method, which is called banded random matrices "BRM", the diagonal elements are drawn from a gaussian sequence that is subject to a zero mean. The order limit of $b = 1$ corresponds to a 1D Anderson tight binding model, which, as shown before, has exponentially localized wave function as solution.It was specifically found that for a finite $N$-dimensional sample with $b < N$, the localization properties of the wave functions scale with the parameter $\Lambda = b^2/N$: for $\Lambda \ll 1$, the BRM eigenvectors show an average exponential decay around some centeral site (localized), while in the opposite limit of $\Lambda \gg 1$, the BRM eigenvectors are extended through out the sample (diffusive). past analysis was performed with the participation number method to study it. it was found that the localization length is given by

$$l(E) = \frac{2}{3}\pi^2\rho^2(E)b^2 \tag{57}$$

where $\rho$ is the DOS, given as

$$\rho(E) = \begin{cases} \frac{1}{4\pi bv^2}(8bv^2 - E^2)^{1/2}, & |E| \leq R_0 = v\sqrt{8b} \\ 0, & |E| > R_0 \end{cases} \tag{58}$$

Here $v^2$ is defined by the variance which is $\sigma^2$ for the gaussian and $W^2/12$ for the box distribution.

# References


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