# **Ex0001:** Travelling Detector of a Momentum State

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**The problem:**

Consider a chain with $N$ sites and hopping amplitude $c$ with periodic boundary conditions, at $t = 0$ the particle is prepared on one of the sites. The detector stays at fixed location and is applied at intervals of $\tau$.

(1) Derive an analytical expression for $p(t)$ where $N = 2$ where the detector is on the second site.

Next steps are done numerically:

(2) For large $N$ plot $p(t)$ where the detector is on the other site of the chain $N/2$ (check both parities of $N$).

The system is now prepared at $t = 0$ in one of the momentum state $|k_m\rangle$

(3) Plot $p(t)$ where the detector is on one of the sites (check both parities of $N$).
(4) The detector is travelling with the same velocity $v_m$ of the prepared state, plot $p(t)$.

**The solution:**

First lets overview the system's design and parameters in order to set a common language for explaining the physical analysis. In order to avoid from redundant information in the sections, when a partial configuration is presented the rest of the parameters are the same as of the last section that they were set.

## 0.1 Apparatus

There are three main components to the system:

### 0.1.1 Ring

Ring is the core component where the physical configuration takes place and the computations of the prepared wave function over time.
The Ring is configured by the following parameters:

- $N = (Integer)$, number of sites.
- $TAUS = (Integer)$, the duration of the experiment in units of $tau$=1.
- $TAU\_INTERVAL = (Integer)$, the resolution of 1[$tau$] interval (partition).
- $HOPPING\_AMPLITUDE = (Float)$, the hopping constant between sites.

### 0.1.2 Sites Detector

Sites Detector component represent an instant of detector on a ring and configured with the following parameters:

- $SITE\_ZERO = (Integer)$, the site number where the particle is prepared at when (t=0).
- $ENABLE\_DETECTOR = (Boolean)$, enable or disable the detector.
- $DETECTOR = (Integer)$, the site number where the detector is placed.
- $DETECTOR\_FREQUENCY = (Integer)$, determines the frequency of detection in units of $[tau]$.

### 0.1.3 Momentum Detector

Momentum Detector component represent an instant of detector on a ring such that the wave function prepared at a specific momentum state $|k_m\rangle$ when $t = 0$. Moreover, the detector can be stationary or travel with the speed of the initial momentum state. Momentum Detector is configured with the following parameters:

- $MOMENTUM\_STATE = (Integer)$, the $m$ index of $|k_m\rangle$ that the wave function prepared at.
- $ENABLE\_DETECTOR = (Boolean)$, enable or disable the detector.
- $DETECTOR = (Integer)$, the site number where the detector is placed at t=0.
- $DETECTOR\_FREQUENCY = (Integer)$, determines the frequency of detection in units of $[tau]$.
- $ENABLE\_DETECTOR\_VELOCITY = (Boolean)$, determines if the detector is stationary or traveling.

The modular design of the system facilitates the preparation of experiments. In order to run an analysis on any desired configuration edit the relevant parameters in the files:

'./analysis/detector_sites_case.py' or './analysis/detector_momentum_case.py

The source code of the project is available in the following github repository:

`https://github.com/karopastal/quantum_states_detector`.

Make sure to read the README.md at the code repository in order to install and execute the experiments properly.

# 1 Two Sites System

First, we construct the most general Hamiltonian for a particle in a two-site system, while making the following assumptions about the system dynamics:
(i) The system is symmetric with respect to reflection.
(ii) The particle can move from site to site with the same hopping amplitude.
After taking those assumptions into account, one gets:

$$\mathcal{H} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

By diagonalizing the Hamiltonian one can determine the eigenvalues and eigenstates of the system:

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

When $|1\rangle$ and $|2\rangle$ are the standard basis unit vectors, while the eigenvalue for each eigenstate is $E_+ = c$ and $E_- = -c$ respectively.
The Hamiltonian in the new basis shall be:

$$\mathcal{H} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

Let us derive an expression for the probability to find a particle in a site, while we prepare the particle in the first site $|1\rangle$ at $t = 0$, as stated in the question, while the detector is placed in the second site.

$$|\psi^{t=0}\rangle = |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

After time $t$ the wave function will be described as such:

$$|\psi^t\rangle = \frac{1}{\sqrt{2}}(e^{-ict}|+\rangle + e^{ict}|-\rangle) = \cos(ct)|1\rangle - i\sin(ct)|2\rangle$$

Therefore, the probability to find the particle in each site $|i\rangle$ at time $t$ is:

$$p(\psi = |1\rangle) = |\cos(ct)|^2 = \frac{1}{2}(1 + \cos(\Omega t))$$

$$p(\psi = |2\rangle) = |i\sin(ct)|^2 = \frac{1}{2}(1 - \cos(\Omega t))$$

Where the frequency of oscillation between the sites is

$$\Omega = E_+ - E_- = 2c$$

The detector collapses the wave function when detecting site 1 at time $t = \tau$. The detection forces the particle to be in site 2.
The wave function in every detection:

$$\psi(n\tau) = |2\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$n = 1, 2, 3...$$

Where $n$ is the number of detection. That means, the wave function in time $n\tau < t < (n+1)\tau$ , evolves as follows:

$$\psi(t) = \frac{1}{\sqrt{2}}(e^{-ict}|+\rangle - e^{ict}|-\rangle)$$

Notice that the wave function is not a differentiable function, due to the collapsing of the wavefunction into a certain site, after each $\tau$ time interval. Hence, the evolution operator is built artificially such that it will collapse the wave function after each detection. We can define the evolution when $n < t/\tau < n + 1$, such as:

$$\hat{S}(n\tau) = \Pi_n \left(\hat{1\!\!1} - |d\rangle\langle d|\right) e^{-i\hat{\mathcal{H}}\tau}$$

Where $d$ is the site of detection. Meaning, we eliminate any evolution from the site of detection, To find how the wave function evolves, we will apply $\hat{S}$ on it:

$$|\phi(n\tau)\rangle = \hat{S}(n\tau)|\psi(t=0)\rangle = \Pi_n \left(\hat{1\!\!1} - |d\rangle\langle d|\right) e^{-i\mathcal{H}\tau}|\psi(t=0)\rangle$$
$$|\psi(t)\rangle = \frac{e^{-i\mathcal{H}(t-n\tau)}|\phi(n\tau)\rangle}{|||\phi(n\tau)\rangle||}$$

$|\phi(n\tau)\rangle$ is the unnormalized wave function evolves at $t = n\tau$, $\hat{S}$ is the new evolution operator, $d$ is the position of the detector and $|\psi\rangle$ is the normalised wave function evolves in time.
The detector collapses the wave function, making the probability to find the particle in the site of the detection zero. The wave function than normalized and starts evolve in time regularly until the next detection.

## 2 N Sites Without Detection

We set the ring with $N = 2$ sites and prepared the particle in site $N = 2$, while the detector is off. The numerical solution for the probability density (to find the particle in each site) as a function of time is as follows:

The parameters configuration in the code for figures 1 and figures 2:

- $TAUS = 50$
- $TAU\_INTERVAL = 20$
- $HOPPING\_AMPLITUDE = 0.09$
- $SITE\_ZERO = 1$
- $ENABLE\_DETECTOR = False$
- $DETECTOR = 1$
- $DETECTOR\_FREQUENCY = 1.$

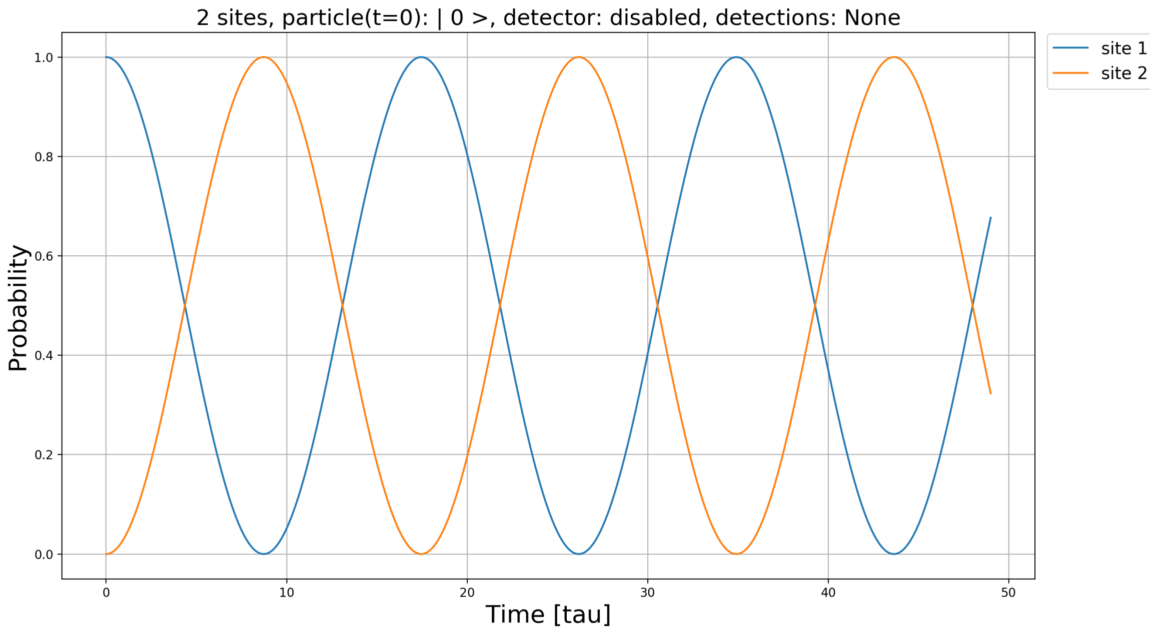


Figure 1: Probability Density, $N = 100$

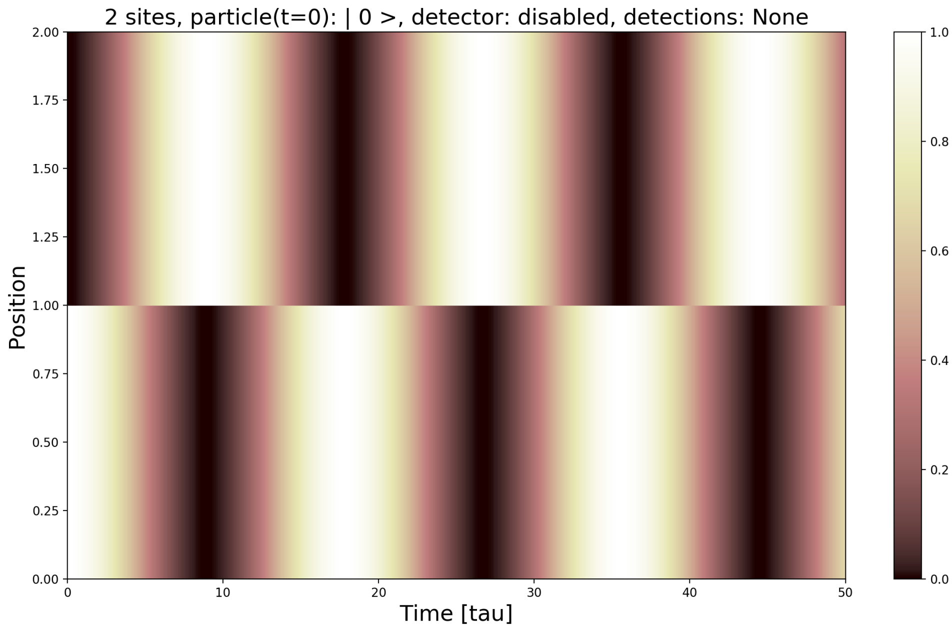


Figure 2: Probability Density, $N = 2$ sites

The parameters configuration for figures 3 and figures 4:

- $TAUS = 50$
- $TAU\_INTERVAL = 20$
- $HOPPING\_AMPLITUDE = 0.09$

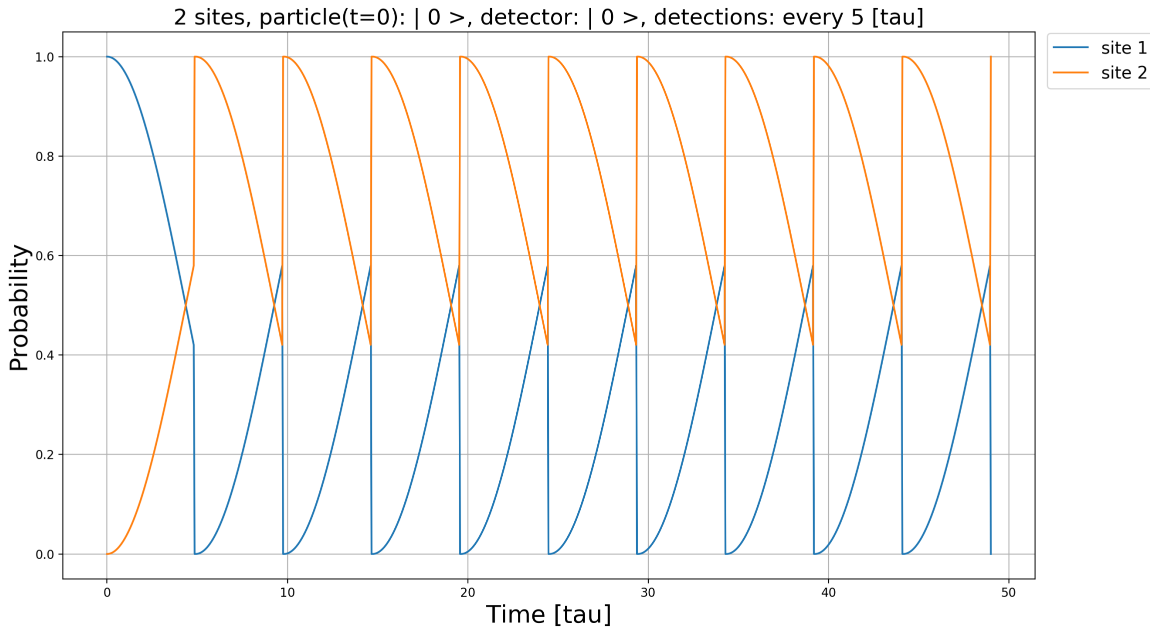


Figure 3: Probability vs. Time, $N = 2$ sites,

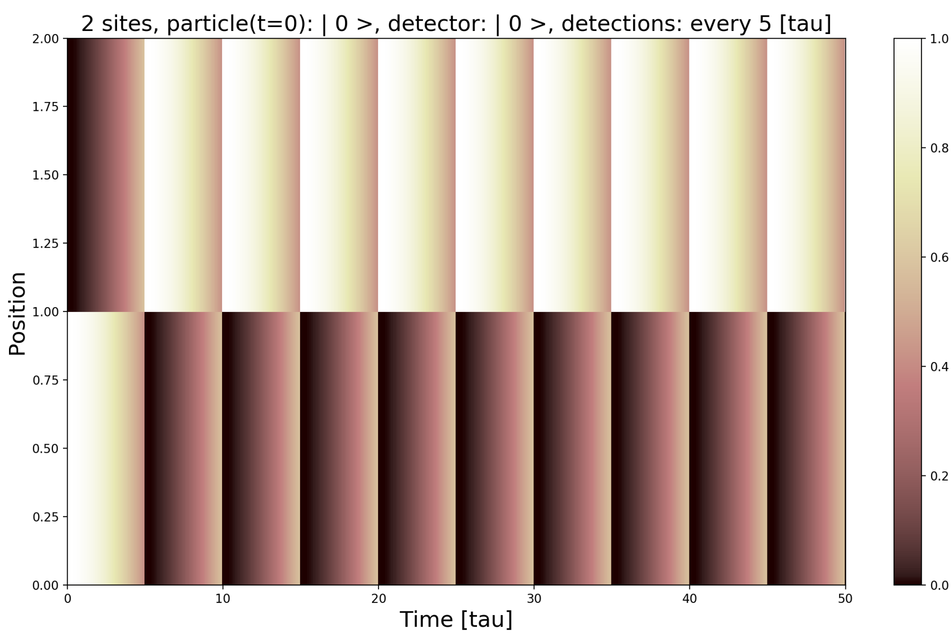


Figure 4: Probability Density, $N = 2$ sites

## 3 N Sites Without Detection

We set the ring with $N$ sites and prepared the particle in site $N = 0$, while the detector is off. Our Hamiltonian is $\hat{\mathcal{H}} = c\hat{D} + c\hat{D}^{\dagger}$ where $D = \sum_i |i\rangle\langle i+1|$ with periodic boundary conditions such that $N+1 = 1$. The numerical solution for the probability density (to find the particle in each site)

as a function of time is as follows:

The parameters configuration in the code for figures 5 and figures 6:

- $TAUS = 100$
- $TAU\_INTERVAL = 20$
- $HOPPING\_AMPLITUDE = 1$
- $SITE\_ZERO = 0$
- $ENABLE\_DETECTOR = False$
- $DETECTOR = 50$
- $DETECTOR\_FREQUENCY = 1.$

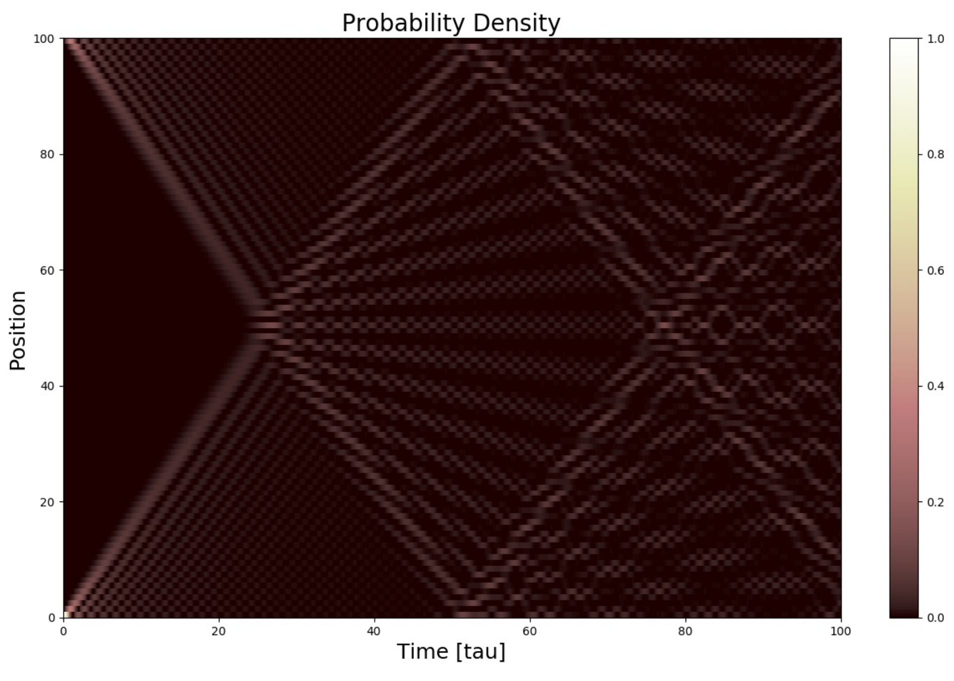


Figure 5: Probability Density, $N = 100$

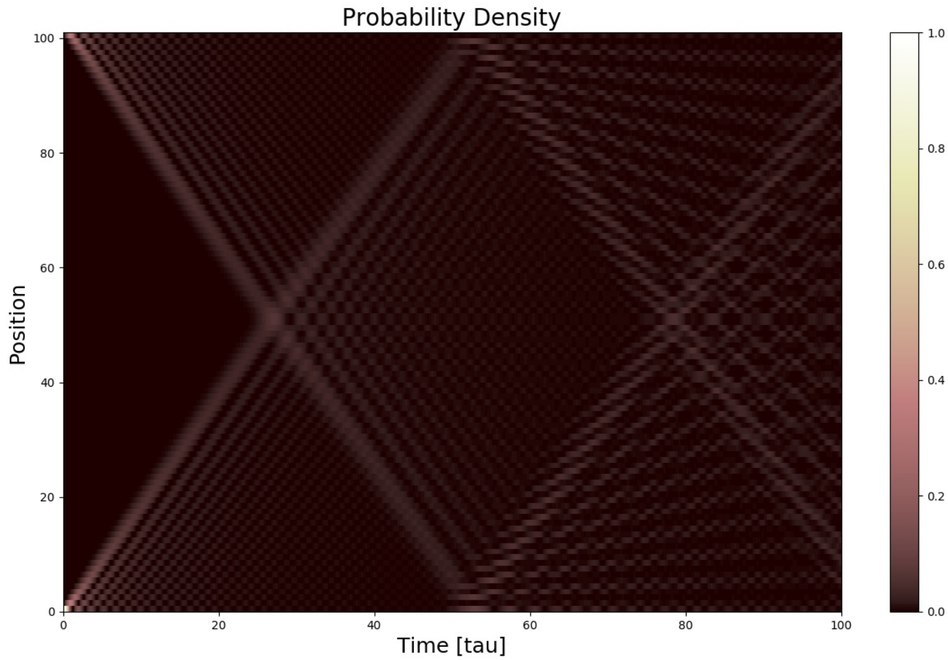


Figure 6: Probability Density, $N = 101$

we can see a difference in the interference pattern between the two configurations of even and odd number of sites. In the odd configuration (Fig 6) the particle is able to make a constructive interference with itself in the site 50, while in the even configuration (Fig 5) the particle reaches site 50 in an opposite phase from each direction, making destructive interference with itself, causing entirely different interference scheme.

## 4 N Sites With Detection

The detector is now able to detect site 50, using the same parameters as in the previous section, (position of the particle is site 0, and the periodic condition, $N+1=1$).
The Hamiltonian remains $\hat{\mathcal{H}} = c\hat{D} + c^*\hat{D}^\dagger$.
The parameters configuration for figures 7 and 8:

- $ENABLE\_DETECTOR = True$

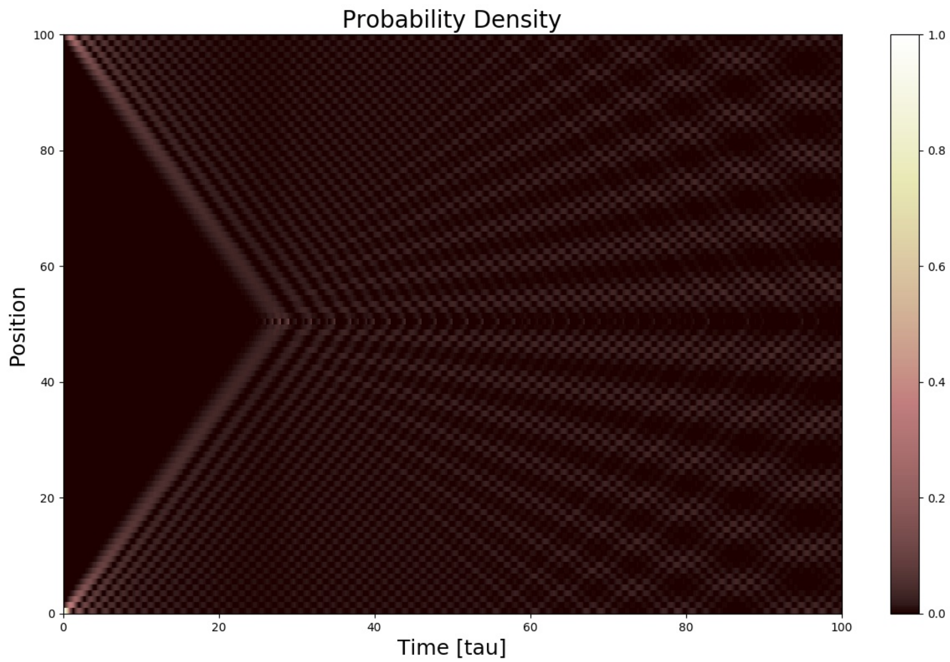


Figure 7: Probability Density, $N = 100$

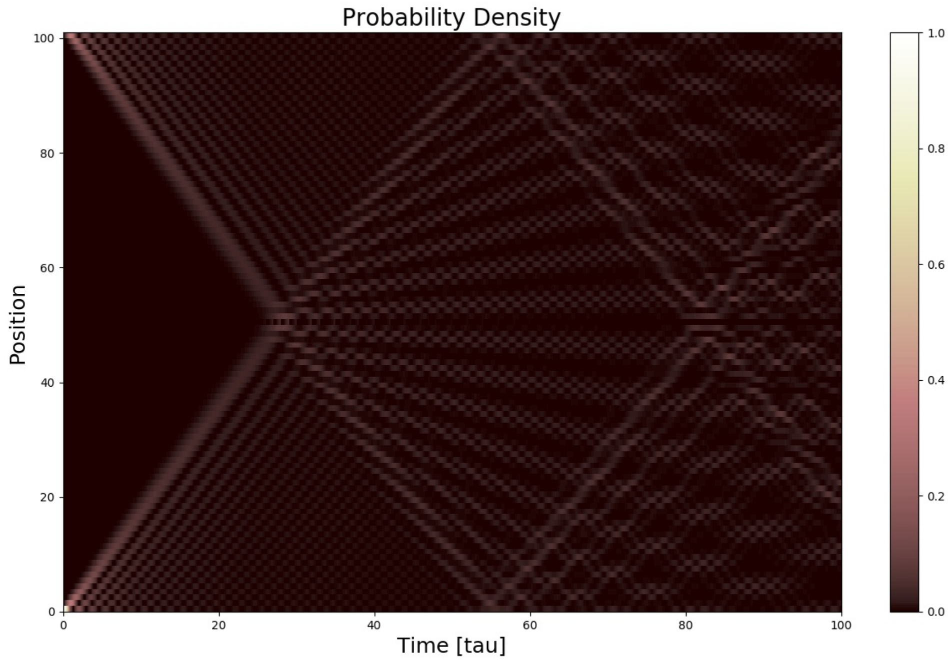


Figure 8: Probability Density, $N = 101$

We can notice yet again a different scheme with each of the configurations. In the even numbered configuration (Fig 7), the detector makes the probability density completely different from the undetected scheme, whereas in the odd numbered system (Fig 8), it seems like the detector detected sites which where in a low probability in the first place, making the scheme looks like an even configuration.

Lets take a look on different configurations: here, the particle has been placed in different sites as follows:

- $SITE\_ZERO = 25$
- $ENABLE\_DETECTOR = True$

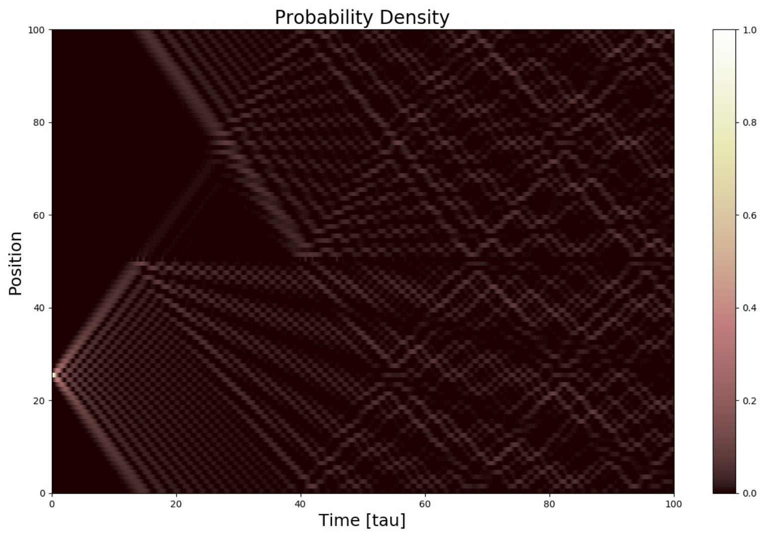


Figure 9: Probability Density, $N = 100$

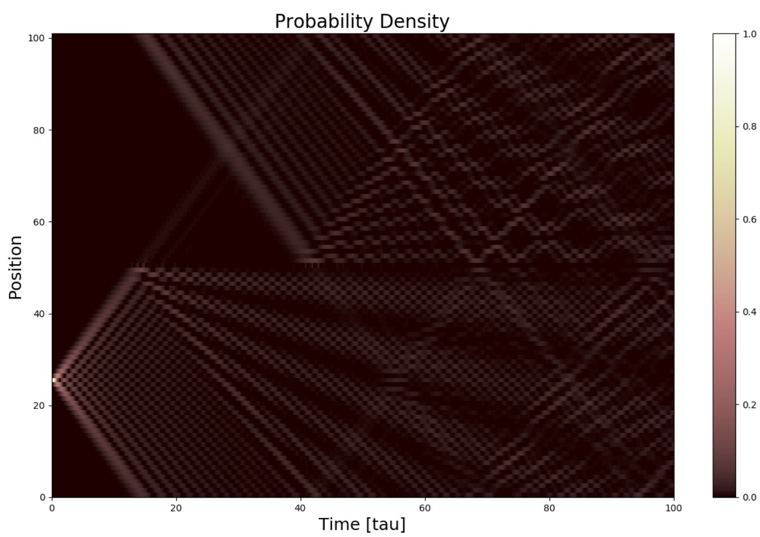


Figure 10: Probability Density, $N = 101$

Here, the number of sites in the ring has been reduced as follows:

- $HOPPING\_AMPLITUDE = 0.5$
- $ENABLE\_DETECTOR = True$
- $DETECTOR = 10$

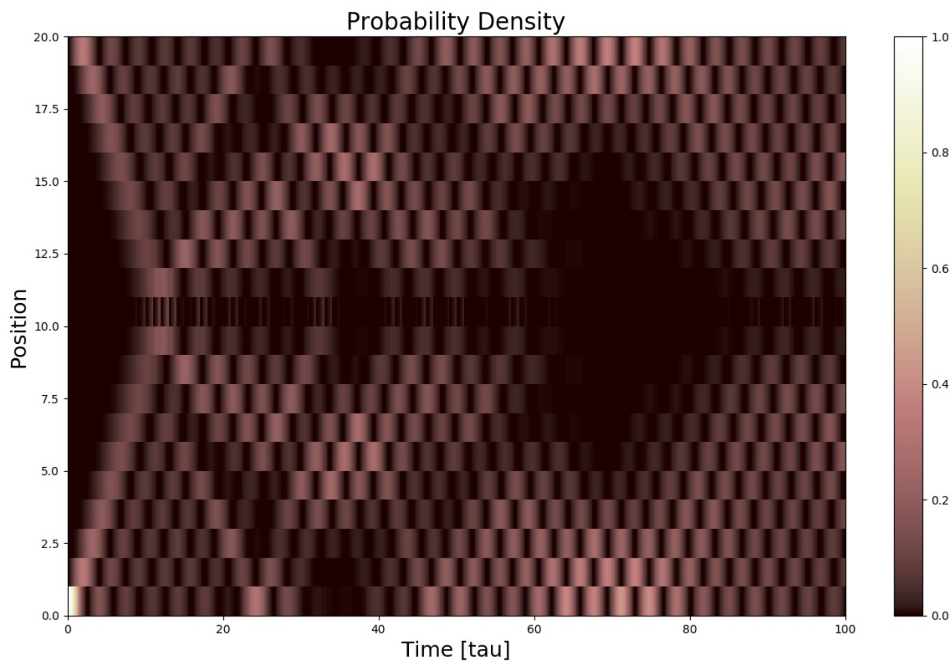


Figure 11: Probability Density, $N = 20$

- $HOPPING\_AMPLITUDE = 0.5$
- $ENABLE\_DETECTOR = True$
- $DETECTOR = 10$
- $DETECTOR\_FREQUENCY = 2.$

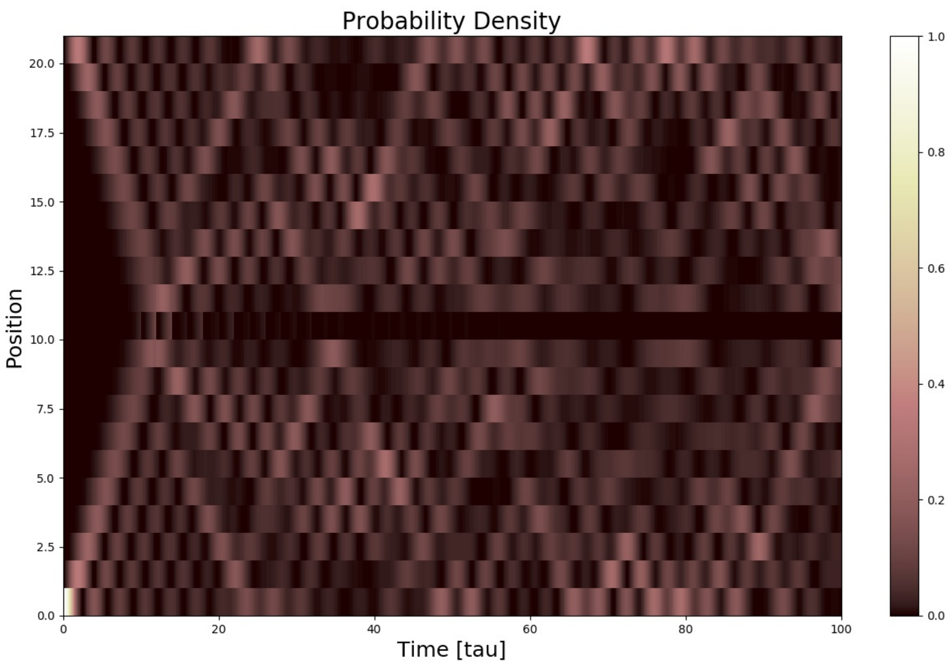


Figure 12: Probability Density, $N = 21$

We can see how the number of sites and the starting position effects the probability density.

# 5 Momentum States With Stationary Detector

We are asked to place the particle in one of the momentum states. The detector will be stationary at site 50.
Our Hamiltonian, again, is $\hat{\mathcal{H}} = c\hat{D} + c\hat{D}^\dagger$ where $D = \sum_i |i\rangle\langle i+1|$ with periodic boundary conditions such that $N+1=1$.
The momentum state is defined as:

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_x e^{ikx} |x\rangle \tag{1}$$

The parameters configuration in the code for the figures in this section:

- $TAUS = 50$
- $TAU\_INTERVAL = 10$
- $HOPPING\_AMPLITUDE = 2$
- $MOMENTUM\_STATE = 50$
- $ENABLE\_DETECTOR = True$
- $DETECTOR = 50$
- $DETECTOR\_FREQUENCY = 5$
- $ENABLE\_DETECTOR\_VELOCITY = False$

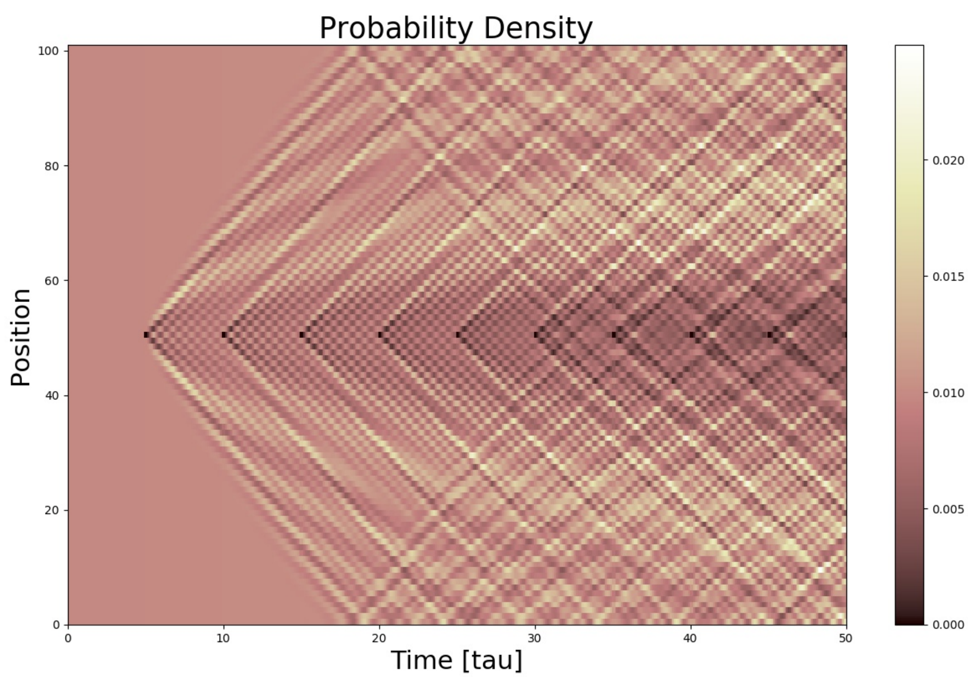


Figure 13: Probability Density, $N = 101$

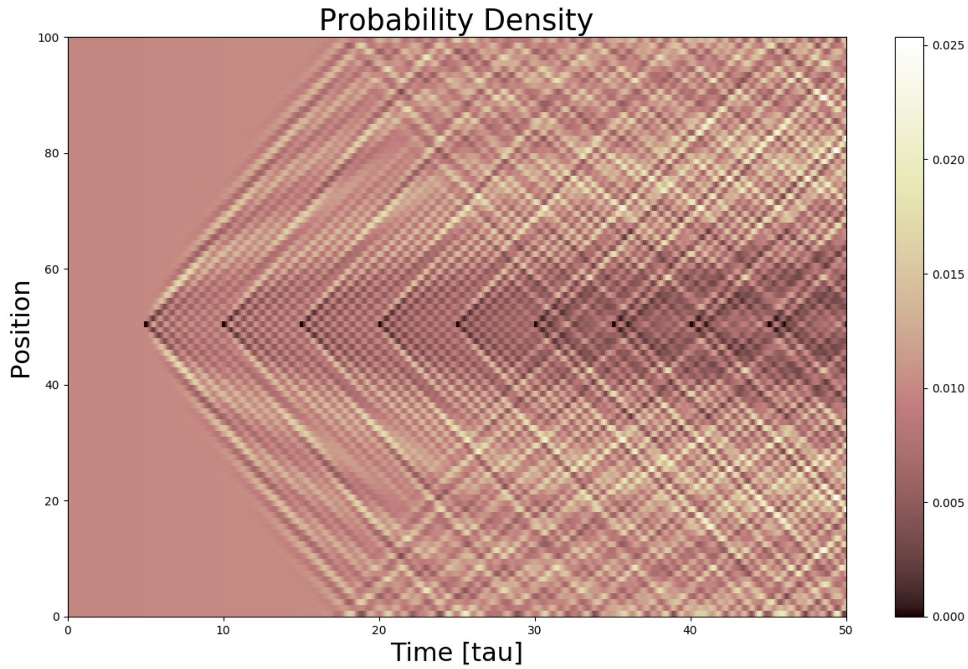


Figure 14: Probability Density, $N = 100$

The particle starts in a momentum state $|k_{50}\rangle$, making the probability to find it in each site equal. As we started the detection, we can see the black sites creating every detection, forcing the particle to change its superposition. While we are asked to show a odd and even sites configurations, (Fig 13 and Fig 14 respectively) we cannot see a substantial difference between the two.

## 6 Momentum States With Traveling Detector

We are now asked to place the particle yet again in a momentum state, while the detector is moving along the ring at a specific velocity, which is the velocity of the eigenstate which the particle is moving. The wanted velocity will be derived as follows:First, defining a Hamiltonian.

$$\hat{\mathcal{H}} = cD + c^* D^{-1} = ce^{-a\hat{p}} + ce^{a\hat{p}} = 2c\cos(a\hat{p}) \tag{2}$$

While $a$ is the distance between each site, we chose $a = 1$ in the numerical solution.Using the definition of the velocity we get:

$$\hat{v} = \frac{\partial \hat{\mathcal{H}}}{\partial p} = -2ca\sin(a\hat{p}) \tag{3}$$

Now, we can find the wanted velocity for each momentum state, using $\hat{v}$:

$$\hat{v}|k\rangle = -2ca\sin(a\hat{p})|k\rangle = -2ca\sin(ak)|k\rangle \tag{4}$$

The parameters configuration in the code for figure 15:

- $TAUS = 100$
- $TAU\_INTERVAL = 5$
- $HOPPING\_AMPLITUDE = 1$
- $MOMENTUM\_STATE = 50$
- $ENABLE\_DETECTOR = True$
- $DETECTOR = 101$
- $DETECTOR\_FREQUENCY = 5$
- $ENABLE\_DETECTOR\_VELOCITY = True$

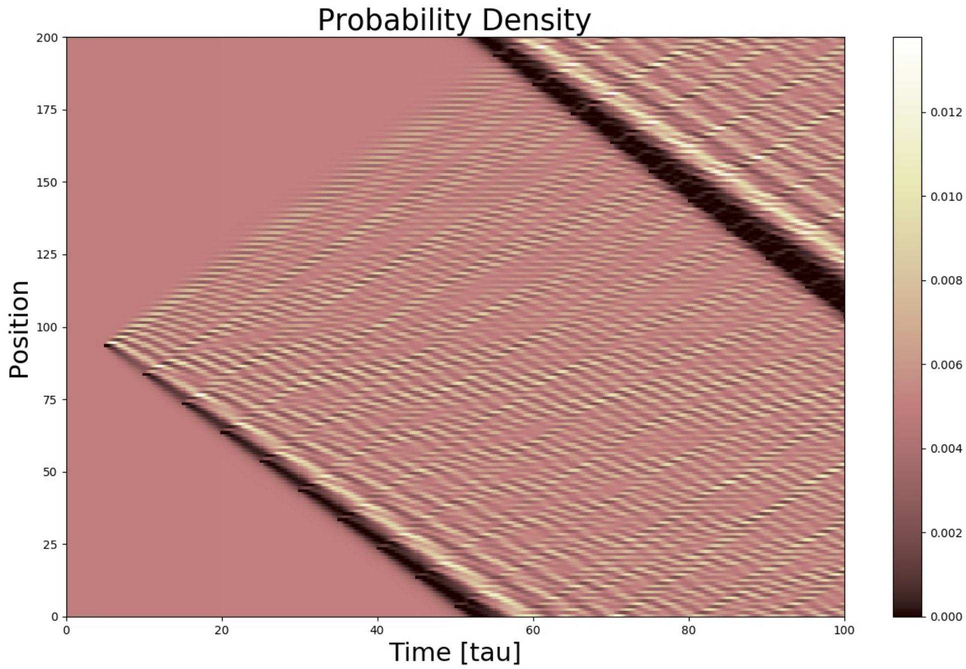


Figure 15: Probability Density, $N = 200$

This interesting configuration, visualises a particle in a momentum state $|k_{50}\rangle$ .The detector travelling in the same velocity of the chosen momentum state. As a result, the detector creates an increasing "hole" in the probability density.