Ex5660: Jaynes–Cummings model

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The problem:

Given the Hamiltonian:

$$\mathcal{H} = \mathcal{H}_{field} + \mathcal{H}_{atom} + \mathcal{H}_{int}$$

Where:

$$\mathcal{H}_{field} = \omega_c \hat{a}^\dagger \hat{a}$$

$$\mathcal{H}_{atom} = \omega_a \frac{\hat{\sigma}_z}{2}$$

$$\mathcal{H}_{int} = \frac{\Omega}{2}(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger)$$

ω_c - the angular frequency of the mode; ω_a - the atomic transition frequency.

(1) Find the constant of motion.
(2) Find the eigenstates and eigenvalues of the Hamiltonian.

The solution:

(1) By setting the zero of energy to the ground state energy of the atom one gets:

$$\mathcal{H}_{atom} = \omega_a |e\rangle\langle e|$$

Thus:

$$\mathcal{H} = \omega_c \hat{a}^\dagger \hat{a} \ + \omega_a |e\rangle\langle e| + \frac{\Omega}{2}(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger)$$

Let us define the "number operator":

$$\hat{N} = |e\rangle\langle e| + \hat{a}^\dagger \hat{a}$$

One can notice that $[\mathcal{H}_{atom}, \hat{N}] = [\mathcal{H}_{field}, \hat{N}] = 0$.

Moreover:

$$\begin{aligned}
[\mathcal{H}_{int}, \hat{N}] &= \frac{\Omega}{2}\Big([\hat{a}\hat{\sigma}_+, |e\rangle\langle e| + \hat{a}^\dagger \hat{a}] + [\hat{a}^\dagger \hat{\sigma}_-, |e\rangle\langle e| + \hat{a}^\dagger \hat{a}]\Big) \\
&= \frac{\Omega}{2}\Big(\hat{a}[\hat{\sigma}_+, |e\rangle\langle e|] + [\hat{a}, \hat{a}^\dagger \hat{a}]\hat{\sigma}_+ + \hat{a}^\dagger[\hat{\sigma}_-, |e\rangle\langle e|] + [\hat{a}^\dagger, \hat{a}^\dagger \hat{a}]\hat{\sigma}_-\Big) \\
&= \frac{\Omega}{2}\Big(-\hat{a}\hat{\sigma}_+ + \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- - \hat{a}^\dagger \hat{\sigma}_-\Big) \\
&= 0
\end{aligned}$$

Thus we deduce:

$$[\mathcal{H}, \hat{N}] = 0$$

Meaning $\hat{N}$ is the constant of motion.

(2) The number operator commutes with the Hamiltonian, thus using its basis of eigenstates $\{|g,0\rangle, |e,0\rangle, |g,1\rangle, ..., |e,n-1\rangle, |g,n\rangle, ...\ \}$ the Hamiltonian takes a blocked-diagonal structure.

$$\mathcal{H} = \begin{bmatrix} H_0 & 0 & ... & ... & ... & ... \\ 0 & \mathcal{H}_1 & 0 & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & 0 & \mathcal{H}_n & 0 & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \end{bmatrix}$$

Where H_0 is a scalar and each $\mathcal{H}_n$ is a 2x2 matrix, we can calculate the elements of the matrix using Dirac's notation:

$$\langle s, n|\mathcal{H}|s', n'\rangle$$

Where s, s' can get values from {e,g}, and n, n' are natural numbers.

We obtain the n^{th} block:

$$\mathcal{H}_n = \begin{bmatrix} \omega_c(n-1)+\omega_a & \frac{\sqrt{n}\Omega}{2} \\ \frac{\sqrt{n}\Omega}{2} & n\omega_c \end{bmatrix}$$

By using the relation:

$$\langle g, n|\mathcal{H}|e, n-1\rangle = \frac{\Omega}{2}\langle g, n|\hat{a}^\dagger\hat{\sigma}_-|e, n-1\rangle + \frac{\Omega}{2}\langle g, n|\hat{a}\hat{\sigma}_+|e, n-1\rangle = \sqrt{n}\frac{\Omega}{2}$$

Next, we can change the Hamiltonian by a constant, $\frac{\omega_c - \omega_a}{2}\hat{I}$, and represent the Hamiltonian as a sum of the following:

$$\mathcal{H}_n = \begin{bmatrix} n\omega_c - \frac{\omega_c-\omega_a}{2} & \frac{\sqrt{n}\Omega}{2} \\ \frac{\sqrt{n}\Omega}{2} & n\omega_c + \frac{\omega_c-\omega_a}{2} \end{bmatrix} = n\omega_c\hat{I} - \frac{\omega_c-\omega_a}{2}\hat{\sigma}_z + \frac{\sqrt{n}\Omega}{2}\hat{\sigma}_x$$

Finally, by diagonalizing the Pauli matrices we get the eigenvalues and eigenstates:

$$E_{n,\pm} = \left(n\omega_c - \frac{\omega_c-\omega_a}{2}\right) \pm \frac{1}{2}\sqrt{(\omega_c-\omega_a)^2 + n\Omega^2}$$

$$|n, +\rangle = \cos\left(\frac{\theta_n}{2}\right)|e, n-1\rangle + \sin\left(\frac{\theta_n}{2}\right)|g, n\rangle$$

$$|n, -\rangle = \cos\left(\frac{\theta_n}{2}\right)|g, n\rangle - \sin\left(\frac{\theta_n}{2}\right)|e, n-1\rangle$$

Where θ_n is defined by the relation: $\tan\theta_n = -\frac{\sqrt{n}\Omega}{\omega_c-\omega_a}$.