ExW11: Magnetic Monopole in a Pyramid

Submitted by: Arbel Schorr and Ben Hevrony

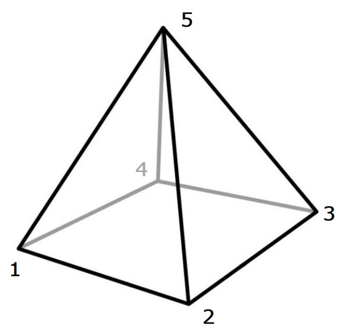
The problem:

Consider a 3D square-based pyramid encapsulating a magnetic monopole. The monopole is placed such that the magnetic flux is equally distributed among the pyramid faces. Define each vertex of the pyramid as a site, transitions are allowed only between sites with a direct connection (no cross transitions) with transition amplitude c. The total flux through the pyramid is Φ, i.e $\frac{\Phi}{5}$ per face. The energy of each site is ε.

(1) Write the Hamiltonian for the system given $\Phi = 0$. Exploit symmetries to find the energy spectrum of the system analytically.
(2) Find a condition on the total flux Φ through the pyramid such that the flux is equally divided between the faces.
(3) Find the energy spectrum for a given Φ which satisfies the condition from item (2)(Computer algebra software is recommended).

The solution:

(1)



Label the sites in counter-clockwise order starting with the front-left site and ending with the fifth as the top vertex. The Hamiltonian in this basis is

$$\hat{\mathcal{H}} = \begin{pmatrix} \varepsilon & c & 0 & c & c \\ c & \varepsilon & c & 0 & c \\ 0 & c & \varepsilon & c & c \\ c & 0 & c & \varepsilon & c \\ c & c & c & c & \varepsilon \end{pmatrix} \xrightarrow[\text{Energy gauge}]{\varepsilon=0} \hat{\mathcal{H}} = \begin{pmatrix} 0 & c & 0 & c & c \\ c & 0 & c & 0 & c \\ 0 & c & 0 & c & c \\ c & 0 & c & 0 & c \\ c & c & c & c & 0 \end{pmatrix}$$

The transformation $|1\rangle \leftrightarrow |2\rangle$, $|3\rangle \leftrightarrow |4\rangle$ is a symmetry of the system and can be represented in this basis by the matrix

$$\hat{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $[\hat{T}, \hat{\mathcal{H}}] = 0$, $\hat{\mathcal{H}}$ is block-diagonal in the eigenbasis of $\hat{T}$. Noting the block formation of Pauli matrices, the diagonalization of $\hat{T}$ is immediate.

$$
\begin{aligned}
|S_1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \\
|A_1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) \\
|S_2\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle + |4\rangle\right) \\
|A_2\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle - |4\rangle\right) \\
|L\rangle &= |5\rangle \qquad (L \text{ because it's lonely...})
\end{aligned}
$$

The Hamiltonian in the new basis can be found by its operation on the new states.

$$
\hat{\mathcal{H}}|S_1\rangle = \frac{1}{\sqrt{2}}\left(\hat{\mathcal{H}}|1\rangle + \hat{\mathcal{H}}|2\rangle\right) = c|S_1\rangle + c|S_2\rangle + \sqrt{2}c|L\rangle
$$

Similarly,

$$
\begin{aligned}
\hat{\mathcal{H}}|S_2\rangle &= c|S_1\rangle + c|S_2\rangle + \sqrt{2}c|L\rangle \\
\hat{\mathcal{H}}|A_1\rangle &= -c|A_1\rangle - c|A_2\rangle \\
\hat{\mathcal{H}}|A_2\rangle &= -c|A_1\rangle - c|A_2\rangle \\
\hat{\mathcal{H}}|L\rangle &= \sqrt{2}c|S_1\rangle + \sqrt{2}c|S_2\rangle
\end{aligned}
$$

Noting the coupling between new basis states, the Hamiltonian in the ordered basis $\{|A_1\rangle, |A_2\rangle, |S_1\rangle, |S_2\rangle, |L\rangle\}$ is now

$$
\hat{\mathcal{H}} = \begin{pmatrix} -c & -c & 0 & 0 & 0 \\ -c & -c & 0 & 0 & 0 \\ 0 & 0 & c & c & \sqrt{2}c \\ 0 & 0 & c & c & \sqrt{2}c \\ 0 & 0 & \sqrt{2}c & \sqrt{2}c & 0 \end{pmatrix}
$$

The block of antisymmetric states can now be treated separately

$$
\hat{\mathcal{H}}^{(A)} = -c \cdot \hat{\mathbb{1}} - c \cdot \hat{\sigma}_x
$$

This block is also immediately diagonalizable (as in the case of $\hat{T}$) with energy eigenvalues:

$$
\begin{aligned}
E_1^{(A)} &= 0 \\
E_2^{(A)} &= -2c
\end{aligned}
$$

Coming back to the symmetric block

$$
\hat{\mathcal{H}}^{(S)} = \begin{pmatrix} c & c & \sqrt{2}c \\ c & c & \sqrt{2}c \\ \sqrt{2}c & \sqrt{2}c & 0 \end{pmatrix}
$$

The transformation $|S_1\rangle \leftrightarrow |S_2\rangle$ represented by

$$
\hat{T}^{(S)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

is a symmetry of this reduced Hamiltonian. Once again since $[\hat{T}^{(S)}, \hat{\mathcal{H}}^{(S)}] = 0$, $\hat{\mathcal{H}}^{(S)}$ is block-diagonal in the eigenbasis of $\hat{T}^{(S)}$, which is trivially diagonalizable. For brevity's sake $\hat{\mathcal{H}}^{(S)} \equiv \hat{h}$.

$$
\begin{aligned}
|S\rangle &= \frac{1}{\sqrt{2}}\left(|S_1\rangle + |S_2\rangle\right) \\
|A\rangle &= \frac{1}{\sqrt{2}}\left(|S_1\rangle - |S_2\rangle\right)
\end{aligned}
$$

The representation of the Hamiltonian in the new basis

$$
\begin{aligned}
\hat{h}|S\rangle &= 2c|S\rangle + 2c|L\rangle \\
\hat{h}|L\rangle &= 2c|S\rangle + 2c|L\rangle \\
\hat{h}|A\rangle &= 0
\end{aligned}
$$

Which yields the third energy eigenvalue,

$$
E_3^{(S)} = 0
$$

In the ordered basis $\{|S\rangle, |L\rangle, |A\rangle\}$

$$
\hat{h} = \begin{pmatrix} 2c & 2c & 0 \\ 2c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

The new reduced block is

$$
\hat{h}^{(S)} = \begin{pmatrix} 2c & 2c \\ 2c & 0 \end{pmatrix} = c \cdot \hat{\mathbb{1}} + 2c \cdot \hat{\sigma}_x + c \cdot \hat{\sigma}_z = c \cdot \hat{\mathbb{1}} + \vec{\Omega} \cdot \hat{\vec{S}}
$$

Where $\hat{\vec{S}}$ is the vector-operator representation of the generators of rotation and $\vec{\Omega} = (4c, 0, 2c)$. Finally the last two energy eigenvalues are given by

$$
E_{4,5}^{(S)} = c \pm \frac{1}{2}\left|\vec{\Omega}\right| = \left(1 \pm \sqrt{5}\right) c
$$

(2) The most general Hamiltonian for this system is:

$$
\hat{\mathcal{H}} = \mathrm{c} \begin{pmatrix}
0 & \mathrm{e}^{-\mathrm{i}\phi_{12}} & 0 & \mathrm{e}^{-\mathrm{i}\phi_{14}} & \mathrm{e}^{-\mathrm{i}\phi_{15}} \\
\mathrm{e}^{\mathrm{i}\phi_{12}} & 0 & \mathrm{e}^{-\mathrm{i}\phi_{23}} & 0 & \mathrm{e}^{-\mathrm{i}\phi_{25}} \\
0 & \mathrm{e}^{\mathrm{i}\phi_{23}} & 0 & \mathrm{e}^{-\mathrm{i}\phi_{34}} & \mathrm{e}^{-\mathrm{i}\phi_{35}} \\
\mathrm{e}^{\mathrm{i}\phi_{14}} & 0 & \mathrm{e}^{\mathrm{i}\phi_{34}} & 0 & \mathrm{e}^{-\mathrm{i}\phi_{45}} \\
\mathrm{e}^{\mathrm{i}\phi_{15}} & \mathrm{e}^{\mathrm{i}\phi_{25}} & \mathrm{e}^{\mathrm{i}\phi_{35}} & \mathrm{e}^{\mathrm{i}\phi_{45}} & 0
\end{pmatrix}
$$

The condition that the flux is identical for each face of the pyramid yields:

$$
\begin{aligned}
-\phi_{12} - \phi_{23} - \phi_{34} + \phi_{14} &= \frac{\Phi}{5} + 2\pi j \\
\phi_{12} + \phi_{25} - \phi_{15} &= \frac{\Phi}{5} + 2\pi k \\
\phi_{23} + \phi_{35} - \phi_{25} &= \frac{\Phi}{5} + 2\pi l \\
\phi_{34} + \phi_{45} - \phi_{35} &= \frac{\Phi}{5} + 2\pi m \\
-\phi_{14} + \phi_{15} - \phi_{45} &= \frac{\Phi}{5} + 2\pi n
\end{aligned}
$$

for some integers j, k, l, m, n. Summing all the equations generates the condition:

$$\Phi = 2\pi N$$

for some integer N.
(3) Utilizing the condition on Φ and demanding equal flux on each face, we can chose transition amplitudes as described in the Hamiltonian below.

$$\hat{\mathcal{H}} = \mathrm{c} \begin{pmatrix} 0 & 1 & 0 & \mathrm{e}^{\frac{2i\phi}{5}} & \mathrm{e}^{\frac{i\phi}{10}} \\ 1 & 0 & \mathrm{e}^{-\frac{2i\phi}{5}} & 0 & \mathrm{e}^{-\frac{i\phi}{10}} \\ 0 & \mathrm{e}^{\frac{2i\phi}{5}} & 0 & 1 & \mathrm{e}^{\frac{i\phi}{10}} \\ \mathrm{e}^{-\frac{2i\phi}{5}} & 0 & 1 & 0 & \mathrm{e}^{-\frac{i\phi}{10}} \\ \mathrm{e}^{-\frac{i\phi}{10}} & \mathrm{e}^{\frac{i\phi}{10}} & \mathrm{e}^{-\frac{i\phi}{10}} & \mathrm{e}^{\frac{i\phi}{10}} & 0 \end{pmatrix}$$

Computer algebra software yields the following energy spectrum:

$$E_1 = c\left[\cos\left(\frac{2\pi N}{5}\right) - \sqrt{\frac{1}{2}\cos\left(\frac{4\pi N}{5}\right) + \frac{9}{2}}\ \right]$$

$$E_2 = c\left[\cos\left(\frac{2\pi N}{5}\right) + \sqrt{\frac{1}{2}\cos\left(\frac{4\pi N}{5}\right) + \frac{9}{2}}\ \right]$$

$$E_3 = -2c\cos\left(\frac{2\pi N}{5}\right)$$

$$E_4 = -2c\sin\left(\frac{2\pi N}{5}\right)$$

$$E_5 = 2c\sin\left(\frac{2\pi N}{5}\right)$$

These expressions for the energy eigenvalues are periodic in N. The common period is 5, therefore the allowed (non-degenerate) values are $N = 0, 1, 2, 3, 4$.