Ex1023: Superradiance

Submitted by: Saeed Younis, Victoria Naipak

The problem:

A toy model for an atom is a symmetric two site system. The two sites are located along the z axis at the points $z = \pm r_0/2$. The hopping amplitude is $\Omega/2$.

1. Write what are the matrix elements of the dipole operator $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$.
2. Express the matrix elements of $\hat{v}$ using those of $\hat{r}$ by exploiting the identity $v = i[H, r]$. The interaction between the EM field and the atom in the dipole approximation can be written as $W = -ev\dot{A}(0)$.Explain why. Write $A(0) = 1/\sqrt{volume}\sum_k ...(a + a^\dagger)$.
3. Find the matrix elements $< k\alpha|A(0)|vacuum >$ for one photon creation and hence find $| < ground, k\alpha|A(0)|excited, vacuum > |^2$.
4. Use Fermi golden rule to find the rate I_0 of spontaneous emission.
5. Assume that you have $N >> 1$ such atoms. What are the possible values of the total J? How many multiplets are there with $j = N/2$, $j = N/2 - 1$ and $j = N/2 - 3$.
6. Assume that at $t = 0$ all the atoms are in the excited state. Prove that the initial rate of radiance is $I = N \times I_0$, while after a transient one obtains the maximum rate $I = (N^2/4)I_0$(superradience).

The solution:

`Should be polished and errors should be corrected.`

1. In our case $r_x = 0$,$r_y = 0$ and $r_z = \frac{1}{2}r_0\sigma_z$ and Hamiltanian of the system is

$$H = \begin{pmatrix} 0 & \Omega/2 \\ \Omega/2 & 0 \end{pmatrix} \tag{1}$$

   Or $H = \frac{1}{2}\Omega\sigma_x$. Thus, so called "dipole matrix elements"' $r_{ab} \neq 0.r_{ab}$ is non diagonal elements, where a,b are eigenstates of σ_x.

2. At first let's define "a = excited" and "b = ground" referring to the two states of the atom the matrix elements of the velocity are derived as follows:

$$\begin{aligned} < ground|\hat{v}|excited > &= i < ground|\ [H, r]\ |excited > \\ &= i(< ground|\ E_a\hat{r}\ |excited > - < ground|\ E_b\hat{r}\ |excited >) \\ &= i(E_a - E_b) < ground|\ \hat{r}\ |excited > \end{aligned} \tag{2}$$

   thus, the final expression for v is

$$v_{ab} = i\omega_{ab}\hat{r}_{ab} \tag{3}$$

   where

   $\omega_{ab} = E_a - E_b$

3. The interaction term is $W = -e\vec{v}\cdot\vec{A}(0)$.

the expression for $A(0)$ is (see lecture notes)

$$A(0) = \frac{1}{\sqrt{volume}}\sum_{k',\alpha'} A_{k',\alpha'}\vec{\epsilon}_{k',\alpha'} \tag{4}$$

$$= \frac{1}{\sqrt{volume}}\sum_{k',\alpha'}\sqrt{\frac{2\pi}{\omega_{k'}}}\vec{\epsilon}_{k',\alpha'}(a^{\dagger}_{k',\alpha'} + a_{k',\alpha'}) \tag{5}$$

where $A_{k',\alpha'}$ are the so called field operators. and can be expressed as

$$A_{k',\alpha'} = \sqrt{\frac{2\pi}{\omega'_k}}(a_{k',\alpha'} + a^{\dagger}_{k',\alpha'}) \tag{6}$$

where $\hat{a}_{k',\alpha'}$ and $\hat{a}^{\dagger}_{k',\alpha'}$ are defined so that

$$\hat{a}_{k',\alpha'}\ |vacuum> = \ 0 \tag{7}$$

and

$$\hat{a}^{\dagger}_{k',\alpha'}\ |n_{k',\alpha'}> = \sqrt{n_{k',\alpha'}+1}\ |n_{k',\alpha'}+1> \tag{8}$$

we now proceed to writing the matrix elements of $< k\alpha|A(0)|vacuum >$

$$< k\alpha|A(0)|vacuum > = < k\alpha|\frac{1}{\sqrt{volume}}\sum_{k',\alpha'}\vec{\epsilon}_{k',\alpha'}\sqrt{\frac{2\pi}{\omega_{k'}}}(a^{\dagger}_{k',\alpha'} + a_{k',\alpha'})|vacuum > \tag{9}$$

since we have a single-energy photon emission the sum over k' is in fact a single term

$$< k\alpha|A(0)|vacuum >= \frac{1}{\sqrt{volume}}\vec{\epsilon}_{k\alpha}\sqrt{\frac{2\pi}{\omega_k}} \tag{10}$$

Now one can see that $| < ground, k\alpha| - e\vec{v}\cdot\vec{A}(0)|excited, vacuum > |^2$ is just

$$\frac{e^2}{volume}\frac{2\pi}{\omega_k}|\vec{v_{ab}}\cdot\vec{\epsilon}_{k\alpha}|^2 \tag{11}$$

We choose the direction of the velocity to coinside with the z-axis

$$\hat{v}_{a,b} \ = i\omega_{ab}\hat{r}_{ab}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

making a general k-vector look like

$$\hat{k} \ = \begin{pmatrix}sin(\theta)cos(\phi)\\ sin(\theta)sin(\phi)\\ cos(\theta)\end{pmatrix}$$

and the two polarization vectors, being perpindicular to it and to each other will look like

$$\hat{\epsilon}_{k02} = \begin{pmatrix} -sin(\phi) \\ cos(\phi) \\ 0 \end{pmatrix}$$

and

$$\hat{\epsilon}_{k01} = \begin{pmatrix} cos(\phi)cos(\theta) \\ sin(\phi)cos(\theta) \\ -sin(\theta) \end{pmatrix}$$

Finally we have

$$| < ground, k\alpha| - e\vec{v}\vec{A}(0)|excited, vacuum > |^2 = \frac{2e^2}{volume}\pi\omega_k r_{ab}^2 sin^2(\theta) \tag{12}$$

4. Fermi Golden rule formula is

$$d\Gamma = 2\pi g(\Omega)d\Omega|W_{ab}|^2 \tag{13}$$

where $g = \frac{L^3k^2}{(2\pi)^3 v_{ab}}$ and $v_{ab} = c$, $\omega_k = ck$

$$d\Gamma = \frac{e^2\omega_k^3 r_0^2 sin^2(\theta)}{8\pi c^3}d\Omega \tag{14}$$

5. Since each atom has two possibilities to occupy a site, the system of N atoms and two sites can be regarded as a system of N spin-halfs.

$$\underbrace{1/2 \otimes 1/2 \otimes 1/2... \otimes 1/2}_{N}$$

the possible values of the "total J" of N spin-halfs are

$$possible\ values\ of\ J\ = \begin{cases} 0\ \ to\ \ N/2 & if\ N\ is\ even \\ 1/2\ \ to\ \ N/2 & if\ N\ is\ odd \end{cases} \tag{15}$$

In Hilbert-space notation, we define a general, normalized, state by

$$|n_a > = \left[\frac{N!}{n_a!n_b!}\right]^{-1/2} \sum_{perm} |\underbrace{a....a}_{n_a}\underbrace{b....b}_{n_b} > \tag{16}$$

where $\frac{N!}{n_a!n_b!}$ is the number of permutations possible in the state

the number of multiplets for given j in an N-particle system is given by the formula

$$number\ of\ multiplets = \frac{N!(2j+1)}{(N/2-j)!(N/2+j+1)!} \tag{17}$$

by direct substitution one has

for $j = N/2$, all the atoms in the exited state, there is only one multiplet.

for $j = N/2 - 1$ there are $N - 1$ multiplets

for $j = N/2 - 3$ there are $\frac{N(N-1)(N-5)}{6}$ multiplets

6. A radiative process involves a decay from a higher energy state to a lower one emitting a photon.
this could be translated as annihilation of a particle in the higher state, creation of a particle in the lower state and creation of a photon with a k vector appropriate to the energy deference. In fock-space notation, this can be written as

$$< n_b + 1, n_a - 1, 1 | {a_b}^{\dagger} a_a {a_{ph}}^{\dagger} | n_b, n_a, 0 > \tag{18}$$

When all the atoms in the excited state,

$$I \;=\; I_0 | < 1, N-1, 1 | a^{\dagger} {a_1}^{\dagger} a_2 | 0, N, 0 > |^2 = |\sqrt{N}|^2 \;=\; N \; I_0 \tag{19}$$

Thus, the decay rate equals $N \times I_0$. after a transient, and assuming that photons escape the system, we'll have a situation of the form

$$I \;=\; I_0 | < N - n_a + 1, n_a - 1, 1 | {a_b}^{\dagger} a_a {a_{ph}}^{\dagger} | N - n_a, n_a, 0 > |^2 = (N - n_a + 1)(n_a) I_0 \tag{20}$$

equaling the derivative of (20) to zero we find it has a maximum when $n_a \;=\; N/2$. in the limit of $N >> 1$ we get the maximal rate of I(superradiance) equals to $(N^2/4)I_0$. Alternative way to get the same result is to consider

$$H^{(N)} = \frac{1}{2} \sum_{N} \sigma_x \equiv J_x \tag{21}$$

And to remember that

$$| < j, m-1 | J_x | j, m > |^2 = j(j+1) - m(m-1) \tag{22}$$

When in our case $j = N/2$.