E1013: Hubbard-Bose Model with Two Sites

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The problem:

Consider a two site system with spinless particles. The system is symmetric to reflections.
The oscillation amplitude is Ω. In Addition, assume that if there are n particles in a site, the interaction is $\sim gn^2$ (explain how to correct this expression in the case of few particles).

1. Write the Hamiltonian.
2. Assuming there is total N particles, show that the interaction energy depends only on the difference of the occupation between the sites.
3. For given N, what is the dimension of the Hilbert space?
4. Define the ladder operator, S_+, that transform a particle from one site to the other.
5. With S_+ define S_x, S_y, S_z, and show that they satisfy the expected commutation relation.
6. Write the Hamiltonian with these operators, and their equation of motion.
7. Explain the difference in the dynamics nature at the limit of strong/weak interaction.

The solution:

1. The Hamiltonian for one particle in a two site system is:

$$\hat{H} = \begin{pmatrix} \varepsilon/2 & c \\ c & -\varepsilon/2 \end{pmatrix} \tag{1}$$

The oscillation amplitude is: $\Omega = \sqrt{(2c)^2 + \varepsilon^2}$ and for symmetric system ε is 0, and hence $c = \Omega/2$.

Using the many body operator, $V = \sum_{k,k'} \hat{a}^\dagger_{k'} V_{k',k} \hat{a}_k$, the Hamiltonian:

$$\begin{aligned} \hat{H} &= \hat{a}^\dagger_1 V_{1,1} \hat{a}_1 + \hat{a}^\dagger_2 V_{2,2} \hat{a}_2 + \hat{a}^\dagger_1 V_{1,2} \hat{a}_2 + \hat{a}^\dagger_2 V_{2,1} \hat{a}_1 \\ &= \frac{\Omega}{2} \hat{a}^\dagger_1 \hat{a}_2 + \frac{\Omega}{2} \hat{a}^\dagger_2 \hat{a}_1 \end{aligned} \tag{2}$$

and the interaction between the particles:

$$\hat{U} = \frac{1}{2} \sum_{\alpha \neq \beta} U_{\alpha\beta,\alpha\beta} \hat{n}_\alpha \hat{n}_\beta + \frac{1}{2} \sum_{\alpha} U_{\alpha\alpha,\alpha\alpha} \hat{n}_\alpha (\hat{n}_\alpha - 1) \tag{3}$$

The first part is the interaction between particles in two different sites, which in our case is 0, and the second part is the interaction between particles in the same site and it is the number of connection between them multiplies by the interaction. In our case it will be $\sim gn^2$. We can use this approximation because when $n \gg 1 \Rightarrow n(n-1) \approx n^2$

Now the Hamiltonian is (here we took ε to be 0):

$$\hat{H} = \frac{\Omega}{2} (\hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1) + \frac{g}{2} (n_1^2 + n_2^2) \tag{4}$$

2. The interaction is: $\frac{g}{2}(n_1^2 + n_2^2)$.
   Define,

$$N \;=\; n_1 + n_2, \qquad n \;=\; n_1 - n_2 \tag{5}$$

   And,

$$n_1 \;=\; \frac{N+n}{2}, \qquad n_2 \;=\; \frac{N-n}{2} \tag{6}$$

   After the substitution, the interaction part becomes:

$$U \;=\; \frac{g}{4}(N^2 + n^2) \tag{7}$$

3. We can evaluate the dimension of the Hilbert space by counting the number of states in the base. The general state looks like: $|n_1, n_2>$, and can get any value as long as $n_1 + n_2 = N$.

$$|n_1, n_2> \;=\; |N, 0>,\; |N-1, 1>,\; ...\; |N-n, n>,\; ...\; |1, N-1>,\; |0, N> \tag{8}$$

   So the Fock space dimension is given by: $dim = N + 1$. The dimension of Hilbert space is the same.

4. The ladder operator S_+, annihilates a particle in site 1 and creates a particle in site 2. The operator S_- does the opposite.

$$\hat{S}_+ \;=\; \hat{a}_2^\dagger \hat{a}_1, \qquad \hat{S}_- \;=\; \hat{a}_1^\dagger \hat{a}_2 \tag{9}$$

5. Accordingly, we'll define:

$$\begin{aligned}
\hat{S}_x &= \frac{1}{2}(S_+ + S_-) \;=\; \frac{1}{2}(\hat{a}_2^\dagger \hat{a}_1 \;+\; \hat{a}_1^\dagger \hat{a}_2) \\
\hat{S}_y &= \frac{1}{2i}(S_+ - S_-) \;=\; \frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2 \;-\; \hat{a}_2^\dagger \hat{a}_1) \\
\hat{S}_z &= \frac{1}{2}(S_- S_+ - S_+ S_-) \;=\; \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 \;-\; \hat{a}_2^\dagger \hat{a}_2) \;=\; \frac{1}{2}(n_1 - n_2)
\end{aligned} \tag{10}$$

   The expected commutation relation is: $[S_i, S_j] = i\epsilon_{ijk} S_k$. We will show that $[S_x, S_y] = iS_z$.

$$\begin{aligned}
[S_x, S_y] &= S_x S_y - S_y S_x \\
&= \frac{1}{2}(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)\frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \;-\; \frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)\frac{1}{2}(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2) \\
&= \frac{i}{2}[\hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 \;-\; \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1] \\
&= \frac{i}{2}[\hat{a}_2^\dagger (1 + n_1) \hat{a}_2 \;-\; \hat{a}_1^\dagger (1 + n_2) \hat{a}_1] \;=\; \frac{i}{2}[n_1(n_2 + 1) \;-\; n_2(n_1 + 1)] \\
&= \frac{i}{2}[n_1 \;-\; n_2] \;=\; iS_z
\end{aligned} \tag{11}$$

   Another way, it can be shown that:

$$S_x \;=\; \sum_{i,j} \hat{a}_i^\dagger \left[\frac{1}{2}\sigma_x\right]_{ij} \hat{a}_j \tag{12}$$

   and the same for S_x, S_y. Hence, S_x, S_y, S_z sustain the expected commutation relation.

6. The Hamiltonian is: $\hat{H} \;=\; \frac{\Omega}{2}(\hat{a}_1^\dagger \hat{a}_2 \;+\; \hat{a}_2^\dagger \hat{a}_1) \;+\; \frac{g}{2}(n_1^2 + n_2^2)$.
   The first part is: ΩS_x.
   The second one is: $\frac{g}{2}(n_1^2 + n_2^2)$, and we will express it in terms of the S_i operators.

$$n_1^2 + n_2^2 \;=\; \frac{1}{2}\left[(n_1+n_2)^2 + (n_1-n_2)^2\right] \;=\; \frac{1}{2}\left[N^2 + (2S_z)^2\right] \tag{13}$$

   The Hamiltonian is:

$$H \;=\; \Omega S_x \;+\; \frac{g}{4}\left[N^2 + (2S_z)^2\right] \;=\; \Omega S_x \;+\; g S_z^2 \;+\; Const \tag{14}$$

   The equations of motion are:

$$\begin{aligned} \dot{S}_x &= i[H, S_x] \qquad (15)\\ &= i\left(\Omega[S_x, S_x] + g[S_z^2, S_x]\right) \\ &= ig[S_z^2, S_x] \;=\; ig(S_z[S_z, S_x] + [S_z, S_x]S_z) \;=\; ig(iS_zS_y + iS_yS_z) \\ &= -g(S_zS_y \;+\; S_yS_z) \end{aligned}$$

   In the same way:

$$\begin{aligned} \dot{S}_y &= -i\Omega[S_x, S_y] + ig[S_z^2, S_y] \;=\; -\,\Omega S_z - g(S_zS_x + S_xS_z) \qquad (16)\\ \dot{S}_z &= i\Omega[S_x, S_z] \;=\; \Omega S_y \end{aligned}$$

7. When $g \ll 1$ the Hamiltonian is ΩS_x and its eigenstates are the same as S_x, which are the symmetric and antisymmetric states.
   When $g \gg 1$ we can treat the Hamiltonian as gS_z^2 with perturbation ΩS_x, and its eigenstates are the same as S_z - the position base.