EX917: Scattering resonances due to poles of the resolvent

Submitted by: Gideon Carmon

The problem: Assume the resolvent has a pole in the lower half of the complex plane in a radial symmetric scattering problem. Find the contribution of the pole to the partial cross section σ_l.

The solution:
We begin by recalling the definition of T matrix:

$$T \equiv V + VG_0V + ... \tag{1}$$

The expansion of G is given by:

$$G = G_0 + G_0VG_0 + ... \tag{2}$$

We can combine both equations and get:

$$T = V + VG_0V + ... = V + VGV \tag{3}$$

note that no approximations are taken so far and the expression of T is exact. We now recall the relation between the phase shift and the T matrix:

$$T_{ll} = e^{i\delta_l}2sin(\delta_l) \tag{4}$$

which gives us the next equation:

$$T_{ll} = e^{i\delta_l}2sin(\delta_l) = V_{ll} + (VGV)_{ll} \tag{5}$$

We can now neglect the first order Born term V_{ll} since its contribution to the resonance scattering is negligible. We're left with:

$$e^{i\delta_l}2sin(\delta_l) \cong (VGV)_{ll} \tag{6}$$

We now turn to write the resolvent as an expansion of the free wave functions:

$$G = \sum_{l,k} \frac{|\phi_{\ell k}\rangle\langle\phi_{\ell k}|}{z - E_k} \tag{7}$$

From now on we limit ourselves to the ℓ channel and therefore omit the ℓ index. Assuming that only one pole is important in the energy range of interest we get Eq.(8) where $|r\rangle$ is a the resonance unnormalized wavefunction.

$$G = \frac{|r\rangle\langle r|}{E - E_r + i\frac{\Gamma_r}{2}} \tag{8}$$

We now take VGV and multiply on both sides with $|\phi^{\ell n}\rangle$ to get $VGV_{\ell\ell}$:

$$(VGV)_{\ell\ell} = \frac{|\langle\phi_{\ell k}|V|r\rangle|^2}{E - E_r + i\frac{\Gamma_r}{2}} \tag{9}$$

By taking Eq.6 and multiply it by its complex conjugate we get:

$$4|sin^2(\delta_l)| \cong \frac{|q_r|^2}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2} \tag{10}$$

were $q_r \equiv |\langle\phi_{\ell k}|V|r\rangle|^2$. We now use the formula derived in Sec[46.3] at the L.N.:

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)|sin(\delta_l)|^2 \tag{11}$$

and we finally get the resonance contribution to the parital cross section:

$$\sigma_l = \frac{\pi(2l+1)}{k^2}\frac{|q_r|^2}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2} \tag{12}$$

By compering the result with the formula derived in class (Sec. [46.9] at the L.N) we get:

$$\frac{\pi(2l+1)}{k^2}\frac{|q_r|^2}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2} = \frac{4\pi(2l+1)}{k^2}\frac{(\Gamma_r/2)^2}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2} \tag{13}$$

which yields:

$$|q_r| = \Gamma_r$$

We can derive this result using the Optical Theorem as well. We begin by using the next Optical Theorem equation (Sec.[45.7]):

$$\sum_{l'}|T_{l'l}|^2 = -2Im[T_{ll}] \tag{14}$$

Since T is diagonal in the free waves base this equation reduces to:

$$|T_{ll}|^2 = -2Im[T_{ll}] \tag{15}$$

From eq.(9) and get:

$$|T_{ll}|^2 = \frac{|q_r|^2}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2} \tag{16}$$

$$-2Im[T_{ll}] = -2\{\frac{-|q_r|(\frac{\Gamma_r}{2})}{(E-E_r)^2+(\frac{\Gamma_r}{2})^2}\} \tag{17}$$

Substituting eq.(16,17)into eq.(15) yields the desired result.