Ex915: Born approximation for scattering from a spherical shell

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The problem:
For the potential $V(r) = V_o R\delta(r-R)$

1. Calculate in Born approximation the quantities $f(\theta)$ and $\frac{d\sigma}{d\Omega}$. Specify limits of validity of your calculation for both high- and low energy scattering, respectively.
2. Calculate the phase shifts δ_l for all partial waves in the approximation that corresponds to the Born approximation.

The solution:

1. The scattering amplitude in the born approximation is given by

$$f^B(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int e^{iq\cdot r}V(\mathbf{r})\,d^3r \tag{1}$$

where m is the mass of the particle and $\mathbf{q} = \mathbf{k_f} - \mathbf{k_i}$ is its momentum transfer. For a spherical symmetric potential, the angular integration can always be performed and (1) reduces to

$$f^B(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty r\sin(qr)V(r)\,dr \tag{2}$$

Substituting in (2) the potential $V(r) = V_o R\delta(r-R)$ we obtain

$$f^B(\theta) = -\frac{2mV_oR^2}{q\hbar^2}\sin(qR) \tag{3}$$

where $q = 2k\sin(\theta/2)$ and $k = \sqrt{2mE/\hbar^2}$. Therefore,

$$\frac{d\sigma}{d\Omega} = |f^B(\theta)|^2 = \frac{4m^2V_o^2R^4}{\hbar^4 q^2}\sin^2(qR) \tag{4}$$

The Born approximation is applicable in case where the scattering potential can be considered as a parturbation, namely, under the condition

$$\left|\int_0^\infty (e^{2ikr}-1)V(r)\,dr\right| \ll \frac{\hbar k}{m} \tag{5}$$

In our problem, this condition of validity can be written as

$$\left|\int_0^\infty (e^{ikr}\sin(kr))V(r)\,dr\right| = V_oR\sin(kR) \ll \frac{\hbar^2 k}{2m} \tag{6}$$

We can now distinguish between two limitings cases that depend on the value of kR:

a) $\frac{2mV_oR\sin(kR)}{k\hbar^2} \leq \frac{2mV_oR}{k\hbar^2} \ll 1$ (high energies)

(7)

b) $\frac{2mV_oR\sin(kR)}{k\hbar^2} \approx \frac{2mV_oRkR}{k\hbar^2} = \frac{2mV_oR^2}{\hbar^2} \ll 1$ (low energies, $kR \ll 1$)

We note that the first condition (kR arbitary)is less restrictive than the second one ($kR \ll 1$). Equation (7.a) indicates that Born approximation is applicable for scattering at sufficiently high energies. Equation (7.b) shows on the other hand that if $kR \ll 1$, then the Born approximation is valid for all velocities $\upsilon = \hbar k/m$ (in both cases one must, of course, consider scattering from a relatively weak and short-ranged potential).

2. The Born approximation corresponds to the case where all the phase shifts are relatively small ($\delta_l \approx \sin(\delta_l) \leq 1$). Thus, we obtain

$$\delta_l = -\frac{\pi m}{\hbar^2}\int_0^\infty rV(r)[J_{l+1/2}(kr)]^2\,dr = -\frac{\pi mR^2V_o}{\hbar^2}[J_{l+1/2}(kR)]^2 \tag{8}$$

Note that using the asymptotic Bessel function expressions

(a) $J_{l+1/2}(x)_{x\to\infty} \to \sqrt{\frac{2}{\pi x}}\sin(x - \pi l/2)$

(9)

(b) $J_{l+1/2}(x)_{x\to 0} \to \sqrt{\frac{2}{\pi}}\frac{x^{l+1/2}}{(2l+1)!!}$

we can recover the condition (7.a).

Substituting expression (9.a) into (8), we find that for arbitary value of kR,

$$\delta_l = -\frac{\pi mR^2V_o}{\hbar^2}\left[\sqrt{\frac{2}{\pi kR}}\sin(kR - \pi l/2)\right]^2 = -\frac{\pi mR^2V_o}{\hbar^2}\frac{2}{\pi kR}\sin^2(kR - \pi l/2) \tag{10}$$

Then from (10) and (7.a) we see that

$$|\delta_l| \leq \frac{\pi mR^2V_o}{\hbar^2}\frac{2}{\pi kR} = \frac{2mV_oR}{k\hbar^2} \ll 1 \tag{11}$$

Thus, the condition $|\delta_l| \ll 1$ for all coincides with (7.a).

Similarly, substituting expression (9.b) into (8) for small value $kR \ll 1$ we find

$$\delta_l = -\frac{\pi mR^2V_o}{\hbar^2}\left[\sqrt{\frac{2}{\pi kR}}\frac{(kR)^{l+1/2}}{(2l+1)!!}\right]^2 = -\frac{\pi mR^2V_o}{\hbar^2}\frac{2}{\pi kR}\frac{(kR)^{2l+1}}{[(2l+1)!!]^2} = -\frac{2mR^2V_o(kR)^{2l}}{\hbar^2[(2l+1)!!]^2} \tag{12}$$

Thus for $l = 0$ and from (7.b)we get

$$|\delta_0| = \frac{2mR^2V_o}{\hbar^2} \ll 1 \tag{13}$$

Then from (12) and (13) for our case $kR \ll 1$ we get

$$|\delta_0| = \frac{2mR^2V_o}{\hbar^2} > |\delta_l| = \frac{2mR^2V_o(kR)^{2l}}{\hbar^2[(2l+1)!!]^2} \tag{14}$$

Finally from (13) and (14) for the small value $kR \ll 1$ we get

$$|\delta_l| < |\delta_0| \ll 1 \tag{15}$$

Thus, the condition $|\delta_l| \ll 1$ coincides with (7.b).