E913: Derivation of the Born approximation for the phase shift

Submitted by: Nadav Barkai

The problem:

In this problem we derive the Born approximation for the phase shift out of the Born approximation for the scattering amplitude $f(\Omega)$.

The solution:

The Born approximation for the scattering amplitude is given by:

$$f(\Omega) = -\frac{m}{2\pi}\langle \mathbf{k}_\Omega | V(r') | \mathbf{k}_0 \rangle = -\frac{m}{2\pi}\int d^3r' \mathrm{e}^{-i\mathbf{k}_\Omega\cdot\mathbf{r}'} V(r') \mathrm{e}^{i\mathbf{k}_0\cdot\mathbf{r}'} \tag{1}$$

Where in our notations:
$\mathbf{k}_0 = k_0\vec{e_z}$
$\mathbf{k}_\Omega = k_0\vec{\Omega}(\theta,\phi)$
$\mathbf{r'} = r'\vec{\Omega}'(\theta',\phi')$
$cos\theta' = \vec{\Omega}'\cdot\vec{e_z}$
$cos\gamma = \vec{\Omega}\cdot\vec{\Omega}'$

Expressing the plane wave $\mathrm{e}^{i\mathbf{k}_0\cdot\mathbf{r}'}$ in terms of partial waves gives

$$\mathrm{e}^{i\mathbf{k}_0\cdot\mathbf{r}'} = \sum_{l'=0}^{\infty} i^{l'}(2l'+1) j_{l'}(k_0r') P_{l'}(cos\theta') = \sum_{l'=0}^{\infty} i^{l'}\sqrt{4\pi(2l'+1)} j_{l'}(k_0r') Y_{l'0}(\theta',\phi') \tag{2}$$

We can also expand the plane wave $\mathrm{e}^{i\mathbf{k}_\Omega\cdot\mathbf{r}'}$ in terms of partial waves

$$\mathrm{e}^{i\mathbf{k}_\Omega\cdot\mathbf{r}'} = \sum_{l=0}^{\infty} i^{l}(2l+1) j_{l}(k_0r') P_{l}(cos\gamma) = 4\pi\sum_{l=0}^{\infty} i^{l} j_{l}(k_0r') \sum_{m=-l}^{l}(-1)^m Y^*_{lm}(\theta,\phi) Y_{lm}(\theta',\phi') \tag{3}$$

Where in the last equation we used the identity

$$P_l(cos\gamma) = \frac{4\pi}{(2l+1)}\sum_{m=-l}^{l}(-1)^m Y^*_{lm}(\theta,\phi) Y_{lm}(\theta',\phi') \tag{4}$$

Inserting (2) and the C.C. of (3) into the scattering amplitude expression gives

$$f(\Omega) = -2m\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sqrt{4\pi(2l'+1)}\int dr'r'^2 d\Omega' (-i)^l (i)^{l'} j^*_l(k_0r') j_{l'}(k_0r') \tag{5}$$

$$(-1)^m Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi') Y_{l'0}(\theta',\phi') V(r')$$

Solving the angular integral we get

$$f(\Omega) = -2m\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sqrt{4\pi(2l'+1)}\int dr'r'^2 (-i)^l (i)^{l'} j_{l'}(k_0r') j^*_l(k_0r') (-1)^m Y_{lm}(\theta,\phi) V(r')\delta_{ll'}\delta_{m0} = \tag{6}$$

$$= -2m\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)} Y_{l0}(\theta,\phi)\int dr'r'^2 j^2_l(k_0r') V(r')$$

where we assume that j_l is a function of a real veriables.
The scattering amplitude can be writen in terms of the phase shift, where for a small phase shift, $\delta_l << 1$, we can write

$$f\left(\Omega\right) = \frac{1}{k_0}\sum_{l=0}^{\infty}\sqrt{4\pi\left(2l+1\right)}Y_{l0}\left(\theta,\phi\right)\mathrm{e}^{i\delta_l}sin\delta_l \approx \frac{1}{k_0}\sum_{l=0}^{\infty}\sqrt{4\pi\left(2l+1\right)}Y_{l0}\left(\theta,\phi\right)\delta_l \tag{7}$$

By comparing the last two equations we finaly get the Born approximation for the phase shift

$$\delta_l^{Born} \approx -2mk_0\int_0^{\infty}dr'r'^2j_l^2\left(k_0r'\right)V(r') = -2\frac{k_0^2}{v_E}\int_0^{\infty}dr'r'^2j_l^2\left(k_0r'\right)V(r') \tag{8}$$