ex:9090 Scattering resonance due to coupling between a dot and a 1D wire

Submitted by: Ben Yellin

The problem:
in this problem you are required to calculate the phase shift $\delta_0(E)$ in a semi 1D scattering problem $x \in [0, \infty]$.
A particle of mass M is sent from infinity with given energy E and returned with a phase shift due to a quantum dot near x_0
The Hamiltonian is given by

$$H = \frac{p^2}{2m} + |0\rangle E_0 \langle 0| + |x_0\rangle \lambda \langle 0| + |0\rangle \lambda \langle x_0|$$

The standard representation of the wave function in the position base is $\Psi \rightarrow (\psi_0, \psi(x))$
the normalization is (as usual) $|\psi_0|^2 + \int_0^\infty |\psi(x)|^2 dx = 1$

1. write down $\langle 0|H|\Psi\rangle$ in the standard representation

2. write down $\langle x|H|\Psi\rangle$ in the standard representation

write down $H\Psi = E\Psi$ using 1,2 and extract a closed equation for $\Psi(x)$ with an effective potential $V(x, E)$

3. write down the effective potential $V(x, E)$

4. find a closed expression for $\delta_0(E)$

5. write down the decay constant Γ using fermi golden rule

6. using 6, find $\delta_0(E)$ with wigner resonance aproximation

7. what is the small paramater that one has to use to show that wigner approximation is consistent

The solution:

1. since Ψ_0 is constant, the first term is zero . the rest of the terms will yield:

$$\begin{aligned}
&\langle 0|0\rangle E_0 \langle 0|\Psi\rangle = E_0 \Psi_0 \\
&\langle 0|x_0\rangle \lambda \langle 0|\Psi\rangle = 0 \cdot \lambda \Psi_0 = 0 \\
&\langle 0|0\rangle \lambda \langle x_0|\Psi\rangle = \lambda \Psi_(x_0) \\
&\langle 0|H|\Psi\rangle = E_0 \Psi_0 + \lambda \Psi(x_0)
\end{aligned}$$

2.

$$\begin{aligned}
&\langle x|\frac{p^2}{2m}|\Psi\rangle = -\frac{\partial^2}{\partial x^2}\psi(x) \\
&\langle x|0\rangle E_0 \langle 0|\Psi\rangle = 0 \cdot E_0 \Psi_0 = 0 \\
&\langle x|x_0\rangle \lambda \langle 0|\Psi\rangle = \delta(x - x_0) \lambda \Psi_0 \\
&\langle x|0\rangle \lambda \langle x_0|\Psi\rangle = 0 \cdot \lambda \Psi(x_0) = 0 \\
&\langle x|H|\Psi\rangle = -\frac{\partial^2}{\partial x^2}\psi(x) + \delta(x - x_0) \lambda \Psi_0
\end{aligned}$$

3.

$$\begin{aligned}
&E_0 \Psi_0 + \lambda \Psi(x_0) = E \Psi_0 \\
&-\frac{\partial^2}{\partial x^2}\psi(x) + \delta(x - x_0) \lambda \Psi_0 = E \psi(x)
\end{aligned}$$

from the first equation we have

$$\Psi_0 = \frac{\lambda}{E - E_0}$$

inserting to the second equation

$$-\frac{\partial^2}{\partial x^2}\psi(x) + \delta(x - x_0)\frac{\lambda^2}{E - E_0}\psi(x) = E\psi(x)$$

hence

$$V(x, E) = \delta(x - x_0)\frac{\lambda^2}{E - E_0}$$

4. using the effective potential, the problems the easiest why to find the phase shift is by matching. the solution for $x < x_0$ is $A\sin(kx)$ (where $k = \sqrt{2ME}$). the solution for $x > x_0$ is $B\sin(kx + \delta)$. Matching the logaritmic derivative at $x = x_0$ will give us

$$k\cot(kx_0) + 2M\frac{\lambda^2}{E - E_0} = k\cot(kx_0 + \delta)$$
$$\delta(E) = -kx_0 + \tan^{-1}\left(\frac{k}{k\cot(kx_0) + \frac{2M\lambda^2}{E - E_0}}\right)$$

5. for a 1D wire of length L (Later L will vanish, so we can take the limit $L \to \infty$) the momentum states in the position base are $\langle x|k\rangle = \sqrt{\frac{2}{L}}\sin(kx)$

$$V_{k,0} = \langle k|V|0\rangle = \langle k|\left(|0\rangle E_0\langle 0| + |x_0\rangle\lambda\langle 0| + |0\rangle\lambda\langle x_0|\right)|0\rangle = \lambda\sqrt{\frac{2}{L}}\sin(kx_0)$$

the decay constant by fermi golden rule is

$$\Gamma = \frac{2\pi}{\Delta_E}|V_{k,0}|^2$$

where the level spacing is $|\Delta_E| = v_E\Delta_k = v_E\frac{\pi}{L}$
so the final result is

$$\Gamma = 2\pi\frac{L}{\pi v_E}\lambda^2\frac{2}{L}\sin^2(kx_0) = \frac{4\lambda^2}{v_E}\sin^2(kx_0)$$

6. using wigner approximation we have

$$\delta_0 \approx -\tan^{-1}\left(\frac{\Gamma/2}{E - E_0}\right) = -\tan^{-1}\left(\frac{2\lambda^2\sin^2(kx_0)}{v_E(E - E_0)}\right)$$

7. approximating 6 we have $\delta_0 \approx -\frac{2\lambda^2 \sin^2(kx_0)}{v_E(E-E_0)}$. taking the result from 4:

$$\begin{aligned}
\delta(E) &= -kx_0 + \tan^{-1}\left(\frac{k}{k\cot(kx_0) + \frac{2M\lambda^2}{E-E_0}}\right) \\
&= -kx_0 + \tan^{-1}\left(\frac{1}{\cot(kx_0) + \frac{2\lambda^2}{v_E(E-E_0)}}\right) \\
&= -kx_0 + \tan^{-1}\left(\frac{\tan(kx_0)}{1 + \underbrace{\frac{2\lambda^2}{v_E(E-E_0)}}_{\text{small parameter}}\tan(kx_0)}\right) \\
&\approx -kx_0 + \tan^{-1}\left(\tan(kx_0) - \frac{2\lambda^2\tan^2(kx_0)}{v_E(E-E_0)}\right) \\
&\approx -kx_0 + \tan^{-1}(\tan(kx_0)) + \frac{1}{\tan^2(kx_0)+1}\left(-\frac{2\lambda^2\tan^2(kx_0)}{v_E(E-E_0)}\right) \\
&= -\frac{2\lambda^2\sin^2(kx_0)}{v_E(E-E_0)}
\end{aligned}$$