EX904: Second order Born approximation (Preface to Kondo)

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The problem:

A particle of mass m is moving within an energy band gap. The particle's energy is E. The band gap consists of energies $E_F < E < E_G$. The density of states within the gap is g. The particle is scattered on a potential V, so that its matrix elements are $V_{kk'} = V_0$ for $|E_k - E_{k'}| < \Delta_0$.

Use Born approximation up to a second order, in order to determine the dependence of the cross-section on the energy.

The solution:

We will use the formula for the cross-section derived in class: $\frac{d\sigma}{d\Omega} = |f(\Omega)|^2$. Where in our case:

$$f(\Omega) = -\frac{m}{2\pi}T_{k_2k_1} \approx -\frac{m}{2\pi}V_{k_2k_1} - \frac{m}{2\pi}\sum_{k'} V_{k_2k'}G_{k'k'}V_{k'k_1}$$

Now let's work out the equation above. The first term is trivial and simply equals to $-\frac{m}{2\pi}V_0$ The second term is a little trickier. First of all, let us write it as an integral and substitute the known form of the resolvent:

$$\sum_{k'} V_{k_2k'}G_{k'k'}V_{k'k_1} \rightarrow \int_{E_F}^{E_G} gdE' \left(V_{k'k_1}\right)^2 \left(\frac{1}{E-E'} - i\pi\delta(E-E')\right) = \int_{E_F}^{E_G} gdE' \left(V_{k'k_1}\right)^2 \frac{1}{E-E'} - i\pi gV_0^2$$

Now we have again a simple term (the second one) and a term that requires a bit of attention. We can distinguish between two cases:
(1) E is not near the band gap edges, so that both $E - \Delta_0$ and $E + \Delta_0$ are within the band gap. In this case the integral is simply taken, within the boundaries $E - \Delta_0$ and $E + \Delta_0$ because those determine the energies for which the potential matrix element is non-zero, so in this case:

$$\int_{E_F}^{E_G} gdE' \left(V_{k'k_1}\right)^2 \frac{1}{E-E'} = gV_0^2 \int_{E-\Delta_0}^{E+\Delta_0} dE' \frac{1}{E-E'} = -gV_0^2 ln|E'-E|_{E-\Delta_0}^{E+\Delta_0} = 0$$

Thus, for the cross-section we have:

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{2\pi}(V_0 - i\pi gV_0^2)\right|^2 = \left(\frac{m}{2\pi}\right)^2 (V_0^2 + \pi^2 g^2 V_0^4)$$

obviously, in case of weak scattering, the second term is negligible.
(2) The second case however, is when the energy is close to one of the boundaries. For example we will take the energy to be close to the upper boundary, so that $E + \Delta_0$ is greater than E_G. Then the upper boundary of the integral will be E_G because the particle cannot be scattered out of the band gap, therefore:

$$\int_{E_F}^{E_G} gdE' \left(V_{k'k_1}\right)^2 \frac{1}{E-E'} = gV_0^2 \int_{E-\Delta_0}^{E_G} dE' \frac{1}{E-E'} = -gV_0^2 ln|E'-E|_{E-\Delta_0}^{E_G} = -gV_0^2 ln\frac{E_G-E}{\Delta_0} = gV_0^2 ln\frac{\Delta_0}{E_G-E}$$

So that the total scattering cross-section in this case is:

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{2\pi}(V_0 - i\pi gV_0^2 + gV_0^2 ln\frac{\Delta_0}{E_G-E})\right|^2 = \left(\frac{m}{2\pi}\right)^2 \left[\left(V_0 + gV_0^2 ln\frac{\Delta_0}{E_G-E}\right)^2 + \pi^2 g^2 V_0^4\right]$$

Note that this cross-section diverges logarithmically as we approach the band gap upper energy. This is probably called someones resonance.
For the lower boundary its exactly the same, just replace $E_G - E$ by $E - E_F$ and the + sign infront of the ln term by a - (the second term again can be neglected if the potential is weak).