Ex:903 - Scattering by a Delta function by Pertubation Theory

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The problem:

In this question, you are being asked to calculate the cross section, of a scattered particle by a 1D delta function.

Asume that a particle with mass M moving inside Energy band, $E_F < E < E_\infty$.
The density of state per unit length in the band, is uniform and equal to ν.
The scattered potential is: $V(r) = u\delta(r)$.

(1) Write down the expression for the return amplitude using T-matrix formalism (first order).
(2) Write down the second order for the return amplitude.
(3) Get the accurate result, by summing an infinit series (assume converging)

The solution:

Opening remarks:

First, we would like to calculate the prefactor of the return amplitude, in 1D.
The free wave looks like:

$\phi^{K_0}(x) = e^{iK_0x}$ [density normalized]

so, the wave function is:

$\Psi = \phi + \Psi^{scatt}$

Therefore:

$\Psi = \phi + G_0^+ T\phi$

$\Psi(x) = e^{iK_0x} + G_0(x\,|\,x')\langle x'|\,T\,|\phi\rangle = e^{iK_0x} - \frac{i}{V_E}e^{iK\,|x-x'|}\langle x'|\,T\,|\phi\rangle$

1. first we need to notice that our problem is one dimensional, which means that for incident wave propagating the $\vec{k}$ direction , the scattered part can only be in $-\vec{k}$ direction. allowing us to drop the vector notation.
the wave function will be in the form:

$$\psi(r) = e^{ik_0r} + f(\Omega)\frac{e^{-ik_0|r|}}{|r|}$$

so we can conclude that the reflection amplitude is $f(\Omega)$

$$f(\Omega) = -\frac{i}{v_E}\langle -k|T(E)|k\rangle$$

taking T to its first order approximation we get $\hat{T} = \hat{V} = u\delta(\hat{x})$

$$f(\Omega) = -\frac{iu}{v_E}\langle -k|\delta(\hat{x})|k\rangle$$

it will be a good idea to insert now a complete basis:

$$f(\Omega) = -\frac{iu}{v_E}\sum_x \langle -k|\delta(\hat{x})|x\rangle\langle x|k\rangle$$

$$f(\Omega) = -\frac{iu}{v_E}\langle -k|0\rangle\langle 0|k\rangle$$

$$f(\Omega) = -\frac{iu}{v_E}$$

2. and now to find the second order $f^{[2]}(\Omega)$

$$f^{[2]}(\Omega) = -\frac{i}{v_E}\langle -k|VG_0V|K\rangle$$

this time we will use a complete basis of the momentum:

$$f^{[2]}(\Omega) = -\frac{i}{v_E}\sum_{k'}\langle -k|VG_0|k'\rangle\langle k'|V|k\rangle$$

$$= \sum_{k'}\langle -k|V|k'\rangle\langle k'|V|k\rangle\frac{1}{E - E_{k'} + i0}$$

already do we know from the previous question, that the first two terms equal u^2 and we get:

$$f^{[2]}(\Omega) = -\frac{iu^2}{v_E}\int\frac{1}{E - E_{k'} + i0}dk'$$

replacing the differential dk' by $dE_{k'}$ and adding the energy distribution we get:

$$f^{[2]}(\Omega) = -\frac{iu^2}{v_E}\int_{E_f}^{E_\infty}\frac{\nu}{E - E_{k'} + i0}dE_{k'}$$

separating the integral by its principal part we get:

$$f^{[2]}(\Omega) = -\frac{iu^2}{v_E}\int_{E_f}^{E_\infty}\left[\frac{\nu}{E - E_{k'}} - i\pi\nu\delta(E - E_{k'})\right]dE_{k'}$$

$$= -\frac{iu^2}{v_E}\nu\left[ln\left(\frac{E - E_f}{E_\infty - E}\right) - i\pi\right]$$

3. Now we shall define:

$g_E = \nu\left[ln\left(\frac{E-E_f}{E_\infty - E}\right) - i\pi\right]$.

T matrix is define by the folowing:

$T_{-k_E,k_E} = u + ug_Eu + \ldots$

$\Rightarrow\ f(\Omega) = -\frac{iu}{V_E}\left[1 + g_Eu + (g_Eu)^2 + \ldots\right]$

It's an infinite geometric series, therefore we get:

$f(\Omega) = -i\frac{u}{V_E}\cdot\frac{1}{1-g_Eu}$