E902: 1D scattering from an inverse squared cosh potential

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The problem:
Scattering from inverse cosh potentail
The solution:

Wrirten style of the problem is not appropriate

(1) Given a 1D potential $V(x)$, use the rst order Born approximation to write the general expression for the transmission and re ection amplitudes.
(Assume the potential is localised around $x = 0$)
(2) Use the result of the previous section to evaluate the transmission and re ection amplitudes for a potential of the form

$$V_0(x) = \frac{A}{cosh^2(qx)}$$

where A, q are constants.
(3) To verify the results to the previous section, you will reduce the problem to a simpler problem you (should) have already solved, and compare the results. Consider the case

$$V_1(x) = \frac{Bq}{cosh^2(qx)}$$

where B is independent of q, (i.e. $A = Bq$). Show that in the limit $q \to \infty$, $V_1(x) \propto \delta(x)$. Use the results of the previous section to evaluate the transmission and re ection amplitudes of $V_1(x)$, and compared them to those of the Delta function.
(4) The Schrödinger equation for the potential

$$V_2(x) = \frac{\hbar^2}{2m}\frac{q^2}{cosh^2(qx)}$$

(i.e. $A = \hbar^2 q^2/2m$) has a relatively simple solution for both bound and free states. The following sections will guide you to the solutions.
(a) Q is a linear operator. Show that if ϕ is an Eigenfunction of $Q^\dagger Q$ with a nonzero eigenvalue, then $Q\phi$ is an eigenfunction of $QQ^\dagger$ with the same eigenvalue.
(b) For

$$Q = \frac{\hbar}{\sqrt{2m}}(-\frac{d}{dx} + f(x))$$

nd $f(x)$ such that $Q^\dagger Q = H_{free} + cosnt.$ and $QQ^\dagger = H_{free} + V_2 + const.$, H_{free} being the hamiltonian for a free particle.
(c) Use the eigenstates of H_{free} to nd the eigenstates of $H_{free} + V_2$
(d) Find the re ection and transmission amplitudes from the explicit expression for the wavefunction.
(e) Find the same amplitudes using the results from section 2, and compare to the results from the previous section.

(f) Prove that $H_{free} + V_2$ has only one bound eigenstate, corresponding to an eigenstate of $QQ^\dagger$ with eigenvalue 0, and nd that bound eigenstate
and eigenvalue.
(g) Find the energy at which the transmission amplitude from section diverges. Compare to the result from the previous section.
The detailed solution:
(1) The forward propagator is

$$G(x-x') = \frac{-i}{2k} e^{ik|x-x'|}$$

Using the First order Born approximation

$$\psi(x) = \phi(x) + \frac{2m}{\hbar^2} \int_{-\infty}^{\infty} G(x-x')V(x')\phi(x')dx' = e^{ikx} + \frac{2m}{\hbar^2} \int_{-\infty}^{\infty} \frac{-i}{2k} e^{ik|x-x'|} V(x') e^{ikx'} dx'$$

The transmission amplitude is equal to the coe cient of the forward-moving wave far from the potential, i.e. $x \to \infty$

$$\lim_{x\to\infty} \psi(x) = \lim_{x\to\infty} \left[e^{ikx} + \left(-\frac{im}{\hbar^2 k} \right) \int_{-\infty}^{\infty} e^{ik|x-x'|} V(x') e^{ikx'} dx' \right] =$$

$$= e^{ikx} - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{ik(x-x')} V(x') e^{ikx'} dx' = e^{ikx} - \frac{im}{\hbar^2 k} e^{ikx} \int_{-\infty}^{\infty} V(x') dx'$$

$$t = 1 - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} V(x') dx'$$

The re ection amplitude is equal to the coe cient of the backward-moving wave far from the potential, i.e. $x \to -\infty$

$$\lim_{x\to-\infty} \psi(x) = \lim_{x\to-\infty} \left[e^{ikx} - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{ik|x-x'|} V(x') e^{ikx'} dx' \right] =$$

$$= e^{ikx} - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} e^{ik(x'-x)} V(x') e^{ikx'} dx' == e^{ikx} - \frac{im}{\hbar^2 k} e^{-ikx} \int_{-\infty}^{\infty} V(x') e^{2ikx'} dx'$$

$$r = -\frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} V(x') e^{2ikx'} dx'$$

(2)

$$t_0 = 1 - \frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} \frac{A \cdot dx'}{\cosh^2(qx')} = 1 - \frac{imA}{\hbar^2 k} \left[\frac{\tanh(qx)}{q} \right]_{-\infty}^{\infty} = 1 - \frac{2imA}{\hbar^2 kq}$$

$$r_0 = -\frac{im}{\hbar^2 k} \int_{-\infty}^{\infty} \frac{Ae^{2ikx'} \cdot dx'}{\cosh^2(qx')} = -\frac{imA}{\hbar^2 k} \left(\int_{-\infty}^{\infty} \frac{\cos(2kx')dx'}{\cosh^2(qx')} + \int_{-\infty}^{\infty} \frac{i\sin(2kx') \cdot dx'}{\cosh^2(qx')} \right)$$

The second term vanishes due to pairity considerations. The rst term can be evaluated using the Fourier transform

$$r_0 = -\frac{imA}{\hbar^2 k} \left(\int_{-\infty}^{\infty} \frac{\cos(2kx')dx'}{\cosh^2(qx')} \right) = -\frac{im}{\hbar^2 k} \frac{2k\pi}{2q^2 \sinh(2k\pi/2q)} = -\frac{2\pi imA}{\hbar^2 q^2 \sinh(\pi k/q)}$$

(3) To prove $\lim_{q\to\infty} V_1(x) \propto \delta(x)$, su ce to show that $V_0(x \neq 0) = 0$ and $\lim_{q\to\infty} \int_{-\infty}^{\infty} V_0(x)dx$ is nite. Verifying the rst condition is trivial, so we'll focus on the second.

$$\lim_{q\to\infty} \int_{-\infty}^{\infty} \frac{Bq}{\cosh^2(qx)} dx = \lim_{q\to\infty} B\left[\tanh(qx)\right]_{-\infty}^{\infty} = \lim_{q\to\infty} 2B = 2B$$

Hence,

$$V_0(x) = 2B\delta(x)$$

According to section 2, the re ection and transmission coe cients are

$$t_1 = \lim_{q\to\infty} \left(1 - \frac{2imBq}{\hbar^2 kq}\right) = 1 - \frac{2imB}{\hbar^2 k}$$

$$r_1 = \lim_{q\to\infty} \frac{-2\pi imBq}{\hbar^2 q^2 \sinh(\pi k/q)} = \lim_{q\to\infty} \frac{-2\pi imB(1/q)}{\hbar^2 \sinh(\pi k/q)} = \lim_{q\to\infty} \frac{-2\pi imB}{\hbar^2(\pi k)\cosh(\pi k/q)} = \frac{-2imB}{\hbar^2 k}$$

(Notice we used L'Hôspital's rule by deriving with respect to $1/q$). The corresponing amplitudes for the delta function are:

$$t_\delta = \frac{1}{1 + 2iB/m\hbar^2 k}$$

$$r_\delta = \frac{1}{im\hbar^2 k/2B - 1}$$

Obviously the two pair of amplitudes di er. That's because the Born approximation holds when the particle's energy is very large. Accordingly, when $B << m\hbar^2 k$

$$t_\delta = \frac{1}{1 + 2iB/m\hbar^2 k} \approx 1 - \frac{2iB}{m\hbar^2 k} = t_1$$

$$r_\delta = \frac{1}{im\hbar^2 k/2B} = \frac{2B}{im\hbar^2 k} \frac{1}{1 - 2B/im\hbar^2 k} \approx -\frac{2iB}{m\hbar^2 k} = r_1$$

(4) a.

$$Q^\dagger Q\phi = \lambda\phi$$

$$QQ^\dagger(Q\phi) = Q(Q^\dagger Q\phi) = \lambda Q\phi$$

b.

$$Q = \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} + f(x)\right)$$

$$Q^\dagger = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} + f(x)\right)$$

$$Q^\dagger Q = \frac{\hbar^2}{2m}\left(\frac{d}{dx} + f(x)\right)\left(-\frac{d}{dx} + f(x)\right) = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} + \frac{df}{dx} + f^2(x)\right)$$

(If you're having di culties with the last calculation, put $\frac{p}{\hbar/i}$ in lieu of $\frac{d}{dx}$). $Q^\dagger Q = H_{free}$i

$$\frac{df}{dx} + f^2(x) = C = \text{const}\,.$$

$$dx = \frac{df}{C - f^2} = \frac{1}{C}\frac{df/C}{1-(f/C)^2} \Rightarrow x - x_0 = \frac{\tanh^{-1}(\frac{f}{C})}{C}$$

$$f(x) = C\tanh(C(x-x_0))$$

To determine C, x_0, we need to calculate $QQ^\dagger$

$$QQ^\dagger = \frac{\hbar^2}{2m}\left(-\frac{d}{dx} + f(x)\right)\left(\frac{d}{dx} + f(x)\right) = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} - \frac{df}{dx} + f^2(x)\right)$$

$$-\frac{df}{dx} + f^2(x) = -\frac{d}{dx}(C\tanh(C(x-x_0))) + (C\tanh(C(x-x_0)))^2 =$$

$$= -C^2\frac{1}{\cosh^2(C(x-x_0))} + C^2\tanh^2(C(x-x_0)) = C^2 - \frac{2C^2}{\cosh^2(C(x-x_0))}$$

Hence, $C = q, x_0 = 0$. Substituting in the expression for $f(x)$

$$f(x) = q\tanh(qx)$$

c.

$$\psi_{free} = e^{ikx}$$

$$\psi_{V_2} = Q\psi_{free} = \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} + q\tanh(qx)\right)e^{ikx} = \frac{\hbar}{\sqrt{2m}}(q\tanh(qx) - ik)$$

d. The wavefunction consists solely of a forward-moving wave, so

$$r_2 = 0$$

The transmission amplitude is equal to the ratio of outgoing to incoming wave coe -cients. Hence,

$$t_2 = \lim_{x\to\infty}\frac{\sqrt{\hbar^2/2m}(q\tanh(qx) - ik)}{\sqrt{\hbar^2/2m}(q\tanh(-qx) - ik)} = \frac{ik - q}{ik + q} = \frac{1 + i(q/k)}{1 - i(q/k)}$$

(You can verify this answer by checking that $|T| = 1$).
e.

$$t_2' = 1 - \frac{2im}{\hbar^2 kq}\left(-\frac{\hbar^2 q^2}{m}\right) = 1 + 2i\frac{q}{k}$$

$$r_2' = -\frac{2\pi im}{\hbar^2 q^2\sinh(\pi k/q)}\left(-\frac{\hbar^2 q^2}{m}\right) = \frac{2\pi i}{\sinh(\pi k/q)}$$

Again, the two pairs of amplitude di er because of the limitation on the Born approx-imation. In the high enrgy regime, i.e. $k >> q$

$$t_2 = \frac{1 - i(q/k)}{1 + (q/k)} \approx 1 + 2i(q/k) = t_2'$$

$$r_2 = \frac{2i\pi}{\sinh(\pi k/q)} \approx 4i\pi e^{-\pi k/q} = 0 + O((q/k)^2)$$

(The last equation can be proved by a Taylor series expansion around $q/k = 0$)
f. The integral over the potential comes out negative, so it's got to have at least one bound state. According to section 4.a., $Q^\dagger Q$ and $QQ^\dagger$ have the same eigenvalue, and since $Q^\dagger Q$ has only positive eigenvalue, so does $QQ^\dagger$. The only exception to that rule in when the eigenvalue is 0. Hence, we must solve

$$QQ^\dagger \psi_b = 0 \Rightarrow Q^\dagger \psi_b = 0$$

$$\left(\frac{d}{dx} + q\tanh(qx)\right)\psi_b = 0$$

$$\frac{d\psi_b}{\psi_b} = -q\tanh(qx)dx$$

$$\ln(\psi_b) = \ln N - \ln(\cosh(qx))$$

$$\psi_b = \frac{N}{\cosh(qx)}$$

N is a normalisation factor. As for the eigenvalue, in section 4.b. we showed that

$$QQ^\dagger = H_{free} + V_2 + \frac{\hbar^2}{2m}q^2$$

$$QQ^\dagger \psi_b = 0 \Rightarrow (H_{free} + V_2)\psi_b = -\frac{\hbar^2}{2m}q^2\psi_b$$

$$E_b = -\frac{\hbar^2}{2m}q^2$$

g. The transmission amplitude diverges when

$$1 - i(q/k) = 0$$

$$k = iq$$

$$E = \frac{\hbar^2}{2m}k^2 = -\frac{\hbar^2}{2m}q^2 = E_b$$