Scattering From A Disc/Poisson's Spot

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The problem:

Given a potential: $V(r) = \begin{cases} V_0 & \sqrt{x^2+y^2} \leq R \\ 0 & else. \end{cases}$
and a plane wave propagating in the z direction:

(1) What is the classical cross section?
(2) Calculate the scattering amplitude $f(\Omega)$ and the cross section in the born approximation. Assume the disc has a finite width.
(3) Write down the scattering state.
(4) What happens if we put a particle detector at $\Omega = 0$?
(5) What happens in the limit of a thin, opaque disc?
(6) Find the total cross section. Assume a small disc.

The solution:

(1) The classical cross section is simply the disc's area: πR^2.

(2) In the born approximation the scattering amplitude is:

$$f(\Omega) = -\frac{m}{2\pi\hbar^2} V(\mathbf{k} = \mathbf{k_\Omega} - \mathbf{k_0}) = -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r$$

(see page 136, formula 836 in the lecture notes).
The integration only has to be performed on the $z = 0$ plane, so we actually have a 2d fourier transform with $k = \sqrt{k_x^2 + k_y^2}$. We choose polar coordinates such that: $\mathbf{k} = k\hat{\mathbf{x}}$ and we get[1]:

$$V(\mathbf{k}) = a \int_{r=0}^{\infty} rV(r) \left\{ \int_0^{2\pi} e^{-ikr\cos(\phi)}\, d\phi \right\} dr = aV_0 \int_0^R rJ_0(kr)\, dr = a\frac{V_0}{k^2} \int_0^{kR} \xi J_0(\xi)\, d\xi$$

Where a is the disc's width. Using the identity:

$$\int^x \xi J_0(\xi) d\xi = xJ_1(x)$$

we get:

$$f(\Omega) = -\frac{V_0 a R m}{\hbar^2} \frac{J_1(kR)}{2\pi k}$$

which has a circular symmetry in $\mathbf{k}$ space, and the cross section is [2]:

$$\sigma(\theta,\phi) = |f(\Omega)|^2 = |\frac{V_0 a R m}{\hbar^2} \frac{J_1(kR)}{2\pi k}|^2$$

(3) The scattering state will be, as usual (page 135, formula 833):

[1] Since we are integrating on $z = 0$, we are only interested in $\mathbf{k}$'s projection on that plane, and therefore can always make this choice of coordinates .

[2] $k = 2k_0 \sin(\frac{\theta}{2})$

$$\psi(\mathbf{r}) = e^{ik_0 z} + f(\Omega)\frac{e^{k_E|r|}}{|r|}$$

(4) Let us cast the expression of the previous section into the form:

$$f(\Omega) = -\frac{V_0 a R^2 m}{\hbar^2}\frac{J_1(kR)}{2\pi kR}$$

Then in limit $k \rightarrow 0$ we have:

$$f(0) = -\frac{V_0 a R^2 m}{\hbar^2}$$

The amplitude of finding a particle in that direction is:

$$\psi(\mathbf{z}) = e^{ikz}(1 - \frac{V_0 a R^2 m}{\hbar^2 z})$$

which is evidently non-zero unless

$$z = \frac{V_0 a R^2 m}{\hbar^2}$$

(5) We see that if $V_0 \rightarrow \infty$ and $a \rightarrow 0$, so that $V_0 a \rightarrow const$, we still have a finite scattering amplitude in this direction. This is called Poisson's Spot.

(6) We calculate:

$$\sigma = \int |f(\Omega)|^2 d\Omega \ = 4\pi(\frac{V_0 a R^2 m}{\hbar^2})^2 \int (\frac{J_1(kR)}{2\pi kR})^2 d\Omega$$

Assuming $kR << 1$:

$$\sigma = 2(\frac{V_0 a R^2 m}{\hbar^2})^2 \int_0^\pi \sin(\theta) d\theta = 4(\frac{V_0 a R^2 m}{\hbar^2})^2$$

Note that if we had used the optical theorem the result would be zero. That's because in order to use the optical theorem one needs a second order approximation to the scattering amplitude, while born's approximation is of first order.