E000: Scattering from a dipole

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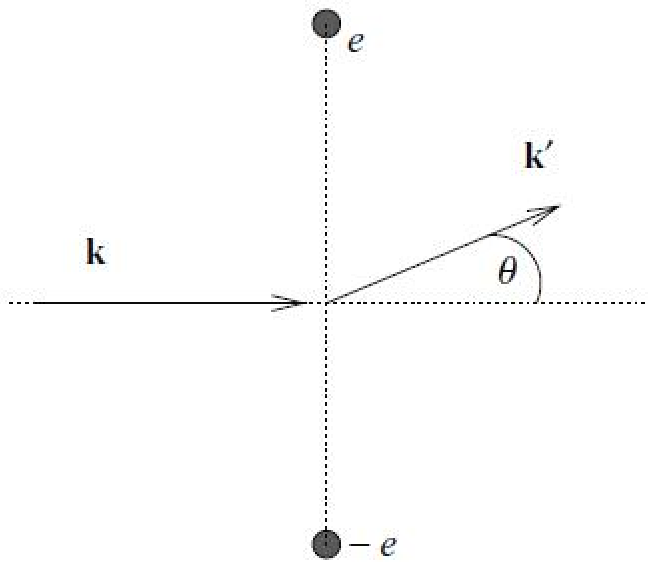

Figure 1: Scattering from a dipole.

The problem:

Consider an electric dipole consisting of two electric charges e and $-e$ at a mutual distance $2a$. Consider also a particle of charge e and mass m with an incident wave vector k perpendicular to the direction of the dipole; see Fig. 1.

1. Calculate the scattering amplitude in the Born approximation. Find the directions at which the differential cross section is maximal.
2. Consider a different system with a target consisting of two arbitrary charges q_1 and q_2 similarly placed. Calculate again the scattering amplitude and the directions of maximal scattering.

The solution:

(1) The potential created by the dipole is

$$V(\mathbf{r}) = -\frac{\mathrm{e}^2}{|\mathbf{r}+\mathbf{a}|} + \frac{\mathrm{e}^2}{|\mathbf{r}-\mathbf{a}|}$$

We have placed the charge $-e$ at $-a$ and the charge e at a. The scattering amplitude in the Born approximation is

$$f_{\mathbf{k}}(\mathbf{k}') = -\frac{4\pi^2\mathrm{m}}{\hbar^2}\langle\mathbf{k}'|\mathrm{V}|\mathbf{k}\rangle = -\frac{4\pi^2\mathrm{me}^2}{\hbar^2}\int\frac{\mathrm{d}^3\mathrm{r}}{(2\pi)^3}\mathrm{e}^{\mathbf{ir\cdot q}}(-\frac{1}{|\mathbf{r}+\mathbf{a}|} + \frac{1}{|\mathbf{r}-\mathbf{a}|}) =$$

$$-\frac{4\pi^2 \mathrm{me}^2}{\hbar^2}\int \frac{\mathrm{d}^3\mathrm{r}}{(2\pi)^3}\mathrm{e}^{\mathbf{ir\cdot q}}\frac{1}{\mathrm{r}}(-\mathrm{e}^{-\mathbf{iq\cdot a}}+\mathrm{e}^{\mathbf{iq\cdot a}})$$

We have denoted $\mathbf{q}\equiv\mathbf{k}-\mathbf{k}'$. The integral involved is

$$\int d^3 r e^{i\mathbf{r}\cdot\mathbf{q}}\frac{1}{r}=2\pi\int_0^\infty dr\, r\int_{-1}^{1} d(cos\Theta)e^{iqrcos\theta}=\frac{4\pi}{q^2}$$

Thus, the scattering amplitude is

$$f_{\mathbf{k}}(\mathbf{k}')=-\frac{2\mathrm{me}^2}{\hbar^2}\frac{1}{\mathrm{q}^2}(-\mathrm{e}^{-\mathbf{iq\cdot a}}+\mathrm{e}^{\mathbf{iq\cdot a}})=-\frac{4\mathrm{ime}^2}{\hbar^2}(\frac{\sin\mathbf{q}\cdot\mathbf{a}}{\mathrm{q}^2})$$

Taking $\mathbf{k}=\mathrm{k}\hat{\mathbf{z}}$ and $\mathbf{a}=\mathrm{a}\hat{\mathbf{x}}$, we have $\mathbf{k}'=\mathrm{k}\sin\theta\hat{\mathbf{x}}+\mathrm{k}\cos\theta\hat{\mathbf{z}}, \mathrm{q}^2=2\mathrm{k}^2(1-\cos\theta)$ and $\mathbf{q}\cdot\mathbf{a}=-\mathrm{ka}\sin\theta$. Thus

$$|f_{\mathbf{k}}(\mathrm{k}')|^2=\frac{4\mathrm{m}^2\mathrm{e}^4}{(\hbar\mathrm{k})^4}\frac{\sin^2(\mathrm{ka}\sin\theta)}{(1-\cos\theta)^2}$$

The cross section becomes maximal when

$$2a\, sin\theta=\frac{\pi}{k}(2n+1)=\frac{\lambda}{2}(2n+1)$$

(2) The scattering amplitude will be

$$f_{\mathbf{k}}(\mathbf{k}')=-\frac{4\pi^2\mathrm{me}}{\hbar^2}\int\frac{\mathrm{d}^3\mathrm{r}}{(2\pi)^3}\mathrm{e}^{\mathbf{ir\cdot q}}\frac{1}{\mathrm{r}}(\mathrm{q}_2\mathrm{e}^{-\mathbf{iq\cdot a}}+\mathrm{q}_1\mathrm{e}^{\mathbf{iq\cdot a}})$$

$$=-\frac{2me}{\hbar^2}\frac{1}{q^2}(q_2 e^{-i\mathbf{q\cdot a}}+\mathrm{q}_1\mathrm{e}^{\mathbf{iq\cdot a}})$$

$$=-\frac{me}{(\hbar k)^2}\frac{(q_1+q_2)\, cos(ka\, sin\theta)+i(q_1-q_2)\, sin(ka\, sin\theta)}{(1-cos\theta)}$$

The differential cross section is

$$|f_{\mathbf{k}}(\mathbf{k}')|^{\mathbf{2}}=\frac{\mathrm{m}^2\mathrm{e}^2}{(\hbar\mathrm{k})^4}\frac{(\mathrm{q}_1+\mathrm{q}_2)^2\cos^2(\mathrm{ka}\sin\theta)+(\mathrm{q}_1-\mathrm{q}_2)^2\sin^2(\mathrm{ka}\sin\theta)}{(1-\cos\theta)^2}$$

$$=\frac{m^2e^2}{(\hbar k)^4}\frac{q_1^2+q_2^2+2q_1q_2cos(2ka\, sin\theta)}{(1-cos\theta)^2}$$

For $q_1q_2>0$, maximal cross section is achieved at

$$2a\, sin\theta=n\frac{2\pi}{k}=n\lambda$$

Wile if $q_1q_2<0$ it is achieved at

$$2a\, sin\theta=(2n+1)\frac{\pi}{k}=(2n+1)\frac{\lambda}{2}$$