E871: Fermi Golden Rule generalization

Submitted by: Eyal Bodinger

The problem:

(1)Write the formal expression for the transition probability $P_t(n|m)$ using the T matrix
$T = V + VG_0V + ...$
(2)Use this expression to derive the Fermi Golden Rule, Explain under which conditions it is valid.
How can it be generalized to higher orders?

The solution:

(1)the transition probability from the m state to n state at time t is

$$P_t(n|m) = |\langle n|U(t)|m\rangle|^2$$

$$\langle n|U(t)|m\rangle = F.T.\left\langle n|G^+|m\right\rangle = F.T.\langle n|G_0 + G_0TG_0|m\rangle = F.T.\langle n|G_0|m\rangle + F.T.\langle n|G_0TG_0|m\rangle$$

$$= F.T.[\frac{\langle n|m\rangle}{E - E_m + i0}] + F.T[\sum_{n'm'} \frac{\langle n|n'\rangle}{E - E'_n + i0} \langle n'|T|m'\rangle \frac{\langle m'|m\rangle}{E - E_m + i0}]$$

the first expression equals zero, we now turn to deal with the second expression.

$$F.T.[\sum_{n'm'} \frac{\langle n|n'\rangle}{E - E'_n + i0} \langle n'|T|m'\rangle \frac{\langle m'|m\rangle}{E - E_m + i0}] = F.T.[\frac{T_{nm}}{(E - E_n + i0)(E - E_m + i0)}]$$

$$= \frac{T_{nm}}{E_m - E_n} F.T.[\frac{1}{E - E_n + i0} - \frac{1}{E - E_m + i0}]$$

In the last equality we assume that T_{nm} is not energy depended.
We recall the Fourier Transform

$$F.T.[\frac{1}{E - E_n + i0}] = \Theta(t)e^{-iE_nt}$$

where $\Theta(t) = 1$ since $t > 0$.

From here we get

$$P_t(n|m) = \frac{|T_{nm}|^2}{(E_m - E_n)^2}|e^{-iE_nt} - e^{-iE_mt}|^2 = \frac{|T_{nm}|^2}{(E_m - E_n)^2}(2 - 2\cos((E_m - E_n)t))$$

$$= \frac{|T_{nm}|^2}{(E_m - E_n)^2}(4\sin^2((E_m - E_n)t/2)) = |T_{nm}|^2t^2\text{sinc}^2((E_m - E_n)t/2)$$

(2) to derive the Fermi Golden Rule we will calculate the probability to stay in the m^{th} state

$$P(t) = 1 - \sum_{n \neq m} P_t(n|m) = 1 - \int \frac{dE}{\Delta} P_t(n|m) = 1 - \int \frac{dE}{\Delta} |T_{nm}|^2 t^2 \mathrm{sinc}^2((E - E_m)t/2)$$

$$= 1 - \frac{2\pi t^2}{\Delta} |T_{nm}|^2 \int \frac{d((E - E_m)t)}{2\pi t} \mathrm{sinc}^2((E - E_m)t/2) = 1 - \frac{2\pi t}{\Delta} |T_{nm}|^2 = 1 - \Gamma t$$

where here we used

$$\int \frac{dx}{2\pi} \mathrm{sinc}^2(x/2) = 1$$

We can see that under the condition $T_{nm} \approx V_{nm}$ we receive the F.G.R.
The Generalization is obtained by considering the higher orders of the T_{nm} expression.