E853: Decay from a Ring

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The problem:

An oxide layer is produced on a golden platform and a "two-dimensinal" golden layer is created above it by evaporation. The undisturbed energy states of an electron within the platform are $|\alpha\rangle$. Their density in the energy space is v. The position states of the partice within the two-dimensional layer are $|x\rangle$. An electron can escape from the two-dimensional layer to the platform. The matrix element is $W_{\alpha,x} = \sigma$.
(1) Write the spatial representation of the imaginary part of the effective hamiltonian of the particle in the two-dimensional layer.
The two dimensional layer is cut into a "one dimensional" ring which its length is L.
(2) write the imaginary part of the hamiltonian that you have found in the momentum representation of the ring.
For the purpose of the transformation to the momentum representation, assume that the ring is composed of a very large number N of sites.
(3) Write the energies and the dacay constants of the "eigenstates" of the particle in the ring. Express your answers by the given (v, σ, N, L) and the electron mass M.

The solution:

`Should be polished.`

(1) The Hamiltonian of the system is:

$$\mathcal{H} = \begin{pmatrix} & & & & \sigma & \cdot & \cdot & \cdot & \sigma \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \mathcal{H}_0^P & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \sigma & \cdot & \cdot & \cdot & \sigma \\ \sigma & \cdot & \cdot & \sigma & E_1^\alpha & & & & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \\ \sigma & \cdot & \cdot & \sigma & & & & & E_n^\alpha \end{pmatrix} = \begin{pmatrix} \mathcal{H}_0^P & 0 \\ 0 & \mathcal{H}_0^Q \end{pmatrix} + \begin{pmatrix} 0 & V^{PQ} \\ V^{QP} & 0 \end{pmatrix}$$

The effective Hamiltonian is:

$$\mathcal{H}_0^P + \Sigma^P$$

Where Σ^P is defined as:

$$\Sigma^P = V^{PQ} \cdot G_0^Q \cdot V^{QP}$$

Since $V_{i,j}^{PQ} = V_{k,l}^{QP} = \sigma$ for every i,j,k,l, and G_0^Q is diagonalized, we get

$$\Sigma_{x,x'}^P = \sum_k \frac{\sigma^2}{E - E_k + i \cdot 0}$$

which can be separated to real and imaginary parts:

$$\sum_k \frac{\sigma^2}{E-E_k} - i\pi\sum_k \sigma^2 \cdot \delta(E-E_k) \equiv \delta E_0 - i\frac{\Gamma_0}{2}$$

Where:

$$\Gamma_0 = 2\pi\sigma^2 \sum_k \delta(E-E_k) = 2\pi\sigma^2\nu$$

Therefore, the imaginary part of the effective Hamiltonian is:

$$\pi\nu\sigma^2 \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}_P$$

(2) Let us transform the imaginary part of the effective Hamiltonian to the momentum space using Dirack's notation:

$$Im[\mathcal{H}_{eff}(k,k')] = \langle k|Im(\mathcal{H}_{eff})|k'\rangle =$$

$$\langle k|(-\pi\nu\sigma^2 \sum_x \sum_{x'} |x\rangle\langle x'|)|k'\rangle = -\pi\nu\sigma^2 \sum_x \langle k|x\rangle \sum_{x'} \langle k'|x'\rangle =$$

$$= -\pi\nu\sigma^2 \sum_x \frac{e^{ikx}}{\sqrt{N}} \sum_{x'} \frac{e^{ik'x'}}{\sqrt{N}}$$

The two summations contain sines and cosines and therefore vanish for very large N except for the the constant values given by $k = 0$. A way to see it is to take the continuum limit. Both of the summations become into integrals of exponents which are delta functions (exactly like Fourier transform of a constant). Hence, we get:

$$Im[\mathcal{H}_{eff}(k,k')] = -\pi\nu\sigma^2 \frac{1}{\sqrt{N}} N\delta_{k,0} \frac{1}{\sqrt{N}} N\delta_{k',0} = -N\pi\nu\sigma^2\delta_{k,0}\delta_{k',0}$$

In matrix form, the Imaginary part of the Hamiltonian is:

$$Im(\mathcal{H}_{eff}) = -N\pi\nu\sigma^2 \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) The possible energies for an electron in a ring are ($\hbar = 1$):

$$E_n = \frac{k^2}{2M} = \frac{(\frac{2\pi}{L}n)^2}{2M} = \frac{2\pi^2}{ML^2}n^2$$

From the imaginary part of the effective Hamiltonian we can see that the only decaying state is at $k = 0$. Its decay constant, Γ_0, is twice the imaginary part of the Hamiltonian:

$$\Gamma_0 = -2N\pi\nu\sigma^2$$