E836: The resolvent of a particle on a ring

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The problem:

you need to find a closed form solution for the sum:
$f(z) = \sum\limits_{n=-\infty}^{\infty} \frac{1}{z-n}$.
assume there is a free particle on a ring with a radius R

(1) what should the Hamiltonian H=h(p) be in order that the resolvent would have poles at energies $E_n = n$
(2) find the resolvent $G(x, x_0)$ for real energies E by solving a differential equation
(3)use the result of the last paragraph in order to find a closed form solution for f(z)

The solution:

(1) a free particle lies on a ring with a radius R (the ring length is L= $2\pi R$)
the wave function of the free particle: $\psi(x) = \frac{1}{\sqrt{L}} e^{ikx}$
from the boundary condition $\psi(0) = \psi(L)$:

$$k_n = \frac{2\pi}{L} n = \frac{n}{R}, \quad n = 0, \pm 1, \pm 2...$$

for $\hat{H}(p) = R\hat{P}$ we get: $\hat{H} \left| K \right> = R\hat{P} \left| K \right> = Rk \left| K \right>$,
where $|K>$ are the momentum states for a free particle.
for this Hamiltonian the resolvent poles are:

$$E_n = < K_n | \hat{H} | K_n > = Rk_n = n$$

(2) by definition, the resolvent is:

$$\text{G(z)} = \frac{1}{z-H}, \quad G(x|x_0) = < x|G(z)|x_0 > = \sum_{n=-\infty}^{\infty} \frac{\psi_n(x)\psi_n^*(x_0)}{z-E_n} = \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{e^{i\frac{n}{R}(x-x_0)}}{z-n}$$

(we see that $G(x|x_0) = G(x - x_0)$).

it's very hard to calculate this sum directly.we can find the resolvent by a different method-
the differential equation for the resolvent:

$$[E - \hat{H}(x)]G(x|x_0) = \delta(x - x_0).$$

let's define $r = (x - x_0)$.
since $\hat{H}(x) = R\hat{P}(x) = -iR\frac{d}{dx}$ we get:

$$(E + iR\frac{d}{dx})G(r) = \delta(r) \rightarrow G(r) = \begin{cases} Ae^{i\frac{E}{R}r}, & L^- > r > 0^+ \\ (A + \frac{i}{R})e^{i\frac{E}{R}r}, & -L^- < r < 0^- \end{cases}$$

this solution fulfills the differential equation $(E + iR\frac{d}{dx})G(r) = 0$ with a jump condition:
$\int\limits_{-\varepsilon}^{\varepsilon} iR\frac{d}{dr}G(r)dr = \int\limits_{-\varepsilon}^{\varepsilon} \delta(r)dr = 1$

now we demand a boundary condition of a ring:

$G(0^-) = G(L^-) \to Ae^{i\frac{E}{R}L} = A + \frac{i}{R}$

$\to A = \frac{i}{R(e^{i\frac{E}{R}L}-1)} = \frac{i}{R(e^{i2\pi E}-1)}.$

we notice that the solution for the differential equation is valid for complex values of E as well as for real values.

(3) we need to calculate the sum: $f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n}$

in paragraph 2 we found that

$G(x-x_0) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{e^{i\frac{n}{R}(x-x_0)}}{z-n}$

therefore $f(z) = LG(0) = 2\pi RG(0)$.
from the differential equation we found that: $G(0^+) = A$, $G(0^-) = A + \frac{i}{R}$ and from the symmetry of the problem:

$G(0) = \frac{G(0^+)+G(0^-)}{2} = A + \frac{i}{2R} = \frac{i}{R(e^{i2\pi z}-1)} + \frac{i}{2R}$

$\to f(z) = \frac{2\pi i}{(e^{i2\pi z}-1)} + \pi i = \pi cot(z\pi)$