E823: Finding the wave function of the eigenstates using the resolvent

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The problem:

Express the wave functions of the eigenstates as integrals over the resolvent in the complex plain.

The solution:

The resolvent is defined in the complex plane as

$$G(z) = \frac{1}{z-H},$$

where H is the hamiltonian.

The matrix elements of the resolvent are

$$G(x|x_0) = \langle x|G(z)|x_0\rangle = \langle x|\frac{1}{z-H}|x_0\rangle = \sum_n \frac{\langle x|n\rangle\langle n|x_0\rangle}{z-E_n} = \sum_n \frac{\psi^n(x)\psi^{n*}(x_0)}{z-E_n}.$$

When taking contour integration around one pole $z = E_m$ only the m-th pole residue will contribute.

$$\oint_{S_m} G(x|x_0)dz = \oint_{S_m} \sum_n \frac{\psi^n(x)\psi^{n*}(x_0)}{z-E_n}dz = 2\pi i \lim_{z\to E_m} \frac{\psi^m(x)\psi^{m*}(x_0)}{z-E_m}(z-E_m),$$

$$\psi^m(x)\psi^{m*}(x_0) = \frac{1}{2\pi i}\oint_{S_m} G(x|x_0)dz,$$

where $\psi^{m*}(x_0)$ is constant.

Hence the wave function is

$$\psi^m(x) = C\oint_{S_m} G(x|x_0)dz.$$

From the condition that the wave function is normalized we can find the constant

$$\int |\psi^m(x)|^2 dx = 1.$$