The propagator of a free particle on a circle

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The problem:

(a) Find the propagator of a free particle on a circle by solving Schrodinger's equation directly.

(b) Repeat (a) but now use Feynman path integral method. Show the results are identical using the Poisson resumation formula:

$$\sum_n e^{-\pi a n^2 + 2\pi i b n} = \frac{1}{\sqrt{a}} \sum_m e^{-\pi \frac{(m-b)^2}{a}}$$

The solution:

(a) The hamiltonian for this problem is

$$\mathcal{H} = -\frac{1}{2MR^2} \frac{\partial^2}{\partial \phi^2}$$

Where R is the radius of the circle, M is the mass of the particle and ϕ is the angle variable. We will use $\hbar = 1$. The boundary condition for the eigenfunctions is

$$\Psi(0) = \Psi(2\pi)$$

So we will guess

$$\Psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$$

and get

$$\mathcal{H}|\Psi_n\rangle = \frac{1}{2MR^2} \frac{1}{\sqrt{2\pi}} (-n^2) e^{-in\phi} = \frac{n^2}{2MR^2} |\Psi_n\rangle = E_n |\Psi_n\rangle$$

Now calculating the propagator is easy. We will directly evaluate the following expression:

$$\langle \phi_f, t|\phi_i, 0\rangle = \langle \phi_f|e^{-i\mathcal{H}t}|\phi_i\rangle = \sum_{n\in\mathbb{Z}} \langle \phi_f|e^{-i\mathcal{H}t}|n\rangle\langle n|\phi_i\rangle = \sum_{n\in\mathbb{Z}} e^{-iE_n t} \langle \phi_f|n\rangle\langle n|\phi_i\rangle = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-iE_n t} e^{-in(\phi_f - \phi_i)}$$

(b) In our circular topology, when a particle moves from one point to another on the circle it can make any integer number of laps around the circle before actually getting from its starting point to the ending point. Therefore, the propagator will be a superposition of propagators with different number of laps. Since we are dealing with a free particle we can use the results obtained in class for the one dimensional free particle propagator using the identification of ϕ with $\phi + 2\pi$.
For the simplicity of calculation we will take $M = 1$ and $\phi_i = 0$ (without loss of generality). Now, using our previous knowledge about the propagator of a free particle (§ 25.4 in the lecture notes) we deduce:

$$\mathcal{U}(\phi_f|0) = \sum_{l\in\mathbb{Z}} \frac{1}{\sqrt{i2\pi t}} \int_0^{\phi_f + 2\pi l} e^{i\mathcal{A}_l[\phi]} = \sum_{l\in\mathbb{Z}} \frac{1}{\sqrt{i2\pi t}} e^{\frac{i}{2t}(\phi_f + 2\pi l)^2}$$

Now we would like to show that the two results are identical using the Poisson resumation formula.

Taking $M = R = \hbar = 1$ and $\phi_i = 0$ the propagator in (a) equals

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{\frac{-itn^2}{2} - in\phi_f}$$

Using the notation $a = \frac{it}{2\pi}$ and $b = -\frac{\phi_f}{2\pi}$ and the resumation formula we will get:

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{\frac{-itn^2}{2} - in\phi_f} = \frac{1}{2\pi}\sum_{n} e^{-\pi a n^2 + 2\pi i b n} = \frac{1}{2\pi}\frac{1}{\sqrt{a}}\sum_{m} e^{-\pi\frac{(m-b)^2}{a}} = \frac{1}{2\pi}\frac{1}{\sqrt{\frac{it}{2\pi}}}\sum_{l\in\mathbb{Z}} e^{-\pi\frac{(l+\frac{\phi_f}{2\pi})^2}{\frac{it}{2\pi}}} = \sum_{l\in\mathbb{Z}}\frac{1}{\sqrt{i2\pi t}} e^{\frac{i}{2t}(\phi_f + 2\pi l)^2}$$

Q.E.D