E816: The propagator of a free particle in the presence of a soft wall

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The problem:

Derive the propagator $U(x_1|x_0)$ for the movement of a particle of mass M during time t in the presence of a potential

$$V(x) = \begin{cases} \frac{1}{2}M\Omega^2 x^2 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

Use Van-Vleck's formula and write explicitly the action and the Morse-Maslow index for every trajectory. For simplicity assume $0 < x_0 < x_1$ and organize your answer stating:

(1) The propagator for short times.
(2) The definition of short times.
(3) The propagator for long times.

The solution:

There are two possible trajectories in which a particle can move from x_0 to x_1. It can either go right straight to x_1 in a constant velocity(free particle). Or it can go left, interact with the harmonic potential and back right to x_1. For obvious reasons they shall be referred to as the *Short* path with action $\mathcal{A}_S(x_1, x_0)$ and the *Long* path with action $\mathcal{A}_L(x_1, x_0)$ respectively.
We will begin by finding $\mathcal{A}_L(x_1, x_0)$. The movement of the particle has 3 domains. (I) is the movement from x_0 to 0 with constant velocity $-v_0$. (II) is the movement left of 0 under the potential. And (III) is the movement from 0 to x_1 with constant velocity v_0.
In domain (II) the particle will act like an harmonic oscillator. During this it will make one half of a period. The time it will take is so one half the time period of an oscillator of frequency Ω.

$$t_{(II)} = \frac{T_{oscillator}}{2} = \frac{\pi}{\Omega}$$

We can now find v_0 by

$$t = t_{(I)} + t_{(II)} + t_{(III)} = \frac{x_0}{v_0} + \frac{\pi}{\Omega} + \frac{x_1}{v_0}$$

$$\Downarrow$$

$$v_0 = \frac{x_0 + x_1}{t - \frac{\pi}{\Omega}}$$

We can now write the classical trajectory for the particle:

In domain (I)
$x(t') = x_0 - v_0 t'$ $\qquad$ $\dot{x}(t') = -v_0$

In domain (II)
$x(t') = -\frac{v_0}{\Omega}\sin\left(\Omega(t' - \frac{x_0}{v_0})\right)$ $\qquad$ $\dot{x}(t') = -v_0\cos\left(\Omega(t' - \frac{x_0}{v_0})\right)$

And in domain (III)
$x(t') = v_0\left(t' - \frac{x_0}{v_0} - \frac{\pi}{\omega}\right)$ $\qquad$ $\dot{x}(t') = v_0$

The action is given by $\mathcal{A}[x] = \int \left(\frac{m\dot{x}^2}{2} - V(x)\right) dt$ and thus:

$$
\begin{aligned}
\mathcal{A}_L(x_1, x_0) = &\int_0^{\frac{x_0}{v_0}} \frac{mv_0^2}{2} dt' \qquad + \\
&\int_{\frac{x_0}{v_0}}^{\frac{x_0}{v_0}+\frac{\pi}{\Omega}} \frac{mv_0^2}{2} \left(\cos^2\left(\Omega(t' - \frac{x_0}{v_0})\right) - \sin^2\left(\Omega(t' - \frac{x_0}{v_0})\right)\right) dt' \qquad + \\
&\int_{\frac{x_0}{v_0}+\frac{\pi}{\Omega}}^{t} \frac{mv_0^2}{2} dt'
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{mv_0^2}{2}\left(\frac{x_0}{v_0}\right) + \frac{mv_0^2}{2}\left(\frac{1}{2} - \frac{1}{2}\right)\left(\frac{\pi}{\Omega}\right) + \frac{mv_0^2}{2}\left(t - (\frac{x_0}{v_0} + \frac{\pi}{\Omega})\right) \\
&= \frac{mv_0^2}{2}\left(t - \frac{\pi}{\Omega}\right)
\end{aligned}
$$

Substituting v_0 we get

$$
\mathcal{A}_L(x_1, x_0) = \frac{m}{2\left(t - \frac{\pi}{\Omega}\right)} (x_0 + x_1)^2
$$

$$
-\frac{\partial^2 \mathcal{A}_L(x_1, x_0)}{\partial x_1 \partial x_0} = -\frac{m}{t - \frac{\pi}{\Omega}}
$$

The Morse-Maslow index for the Long path is 1 as there is a single conjugate point in its trajectory.

For a free particle moving in the Short path we know

$$
\mathcal{A}_S(x_1, x_0) = \frac{m}{2t} (x_0 - x_1)^2
$$

$$
-\frac{\partial^2 \mathcal{A}_S(x_1, x_0)}{\partial x_1 \partial x_0} = \frac{m}{t}
$$

The Morse-Maslow index for the Short path is 0 as there are no conjugate points in its trajectory.

Now we shall consider the possible trajectories for various given times of motion.

(1) For $t < \frac{\pi}{\Omega}$ the only possible trajectory is the Short path. By Van-Vleck's formula we get the propagator for short times:

$$
U^S(x_0|x_1) = \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} e^{i\frac{m}{2t}(x_0 - x_1)^2}
$$

(2) A short time is $\{t | t < \frac{\pi}{\Omega}\}$

(3) For longer times there are two available trajectories and so the propagator becomes:

$$
\begin{aligned}
U^L(x_0|x_1) &= \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} e^{i\frac{m}{2t}(x_0 - x_1)^2} + \left(\frac{m}{2\pi i (t - \frac{\pi}{\Omega})}\right)^{\frac{1}{2}} e^{i\frac{m}{2(t - \frac{\pi}{\Omega})}(x_0 + x_1)^2 - i\frac{\pi}{2}} \\
&= \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} e^{i\frac{m}{2t}(x_0 - x_1)^2} - i\left(\frac{m}{2\pi i (t - \frac{\pi}{\Omega})}\right)^{\frac{1}{2}} e^{i\frac{m}{2(t - \frac{\pi}{\Omega})}(x_0 + x_1)^2}
\end{aligned}
$$