Exercise 815: Propogator of free particle, with a rebound wall

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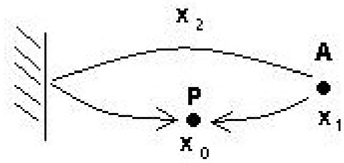
Problem

Consider a free particle in one dimension, moving from point A to point P, the particle can move in two different paths : x_1-the direct path, and x_2-to the wall and back, as described in the picture.



Calculate $U(x|x_0)$, what is the meaning of the answer?

Solution

We can calculate each path separately and then combine them together.
Therefore :

$$U(x|x_0) = U^{(1)} + U^{(2)}$$

The first path (the direct path), is simply the path of free particle, between two points x_0 and x_1 :

The propogator defined as:

$$U(x|x_0) = \sum_{cl} (\frac{1}{\imath 2\pi\hbar})^{\frac{d}{2}} \sqrt{det(-\frac{\partial^2 \mathcal{A}_{cl}}{\partial x \partial x_0})} e^{\imath\frac{\pi}{2}\nu^{\alpha}} e^{\frac{\imath}{\hbar}\mathcal{A}(x|x_0)}$$

Where $-\imath\frac{\pi}{2}\nu^{\alpha}$ called **Maslov/Morse index**, and ν^{α} - number of conjugate points up to time t.

*** Explanation:** ---
Consider the linearized equation of motion for $x(t) = x_{cl}(t) + \delta x(t)$,which is :

$$m\delta\ddot{x} + V^{''}(x_{cl})\delta x = 0$$

A conjugate point (in time) is defined as the time when the linearized equation has a non-trivial solution.

With this rule we expect that in case of a reflection from a wall (semiclasical), the Morse index is **+1** for each collision, but the correct result in case of "hard" wall is **+2**. This is implied by **Dirichlet** boundary conditions. Note, that it would be **+0** in case of **Neumann** boundary conditions.

In other words, every time the particle "recoils", it has phase shifted, so the posible values of this term can be:

$$e^{\imath \frac{\pi}{2}\nu} = \begin{cases} 1 & \text{for } \nu = 0, 4, ..., 4n \\ \imath & \text{for } \nu = 1, 5, ..., 4n+1 \\ -1 & \text{for } \nu = 2, 6, ..., 4n+2 \\ -\imath & \text{for } \nu = 3, 7, ..., 4n+2 \end{cases}$$

Now, We have to find $\mathcal{A}(x)$ for the free particle, i.e.

$$\mathcal{A}(x) = \int_0^t \frac{1}{2} m\dot{x}^2 dt'.$$

Finding path : one dimensional free particle

$$x_{cl} = x_0 + vt' = x_0 + \frac{x_1 - x_0}{t} t'$$

Computing the Action :

$$\dot{x} = \frac{x_1 - x_0}{t} \Longrightarrow \mathcal{A}_{cl}(x_1, x_0) = \int_0^t \frac{1}{2} m (\frac{x_1 - x_0}{t})^2 dt' = \frac{m}{2t}(x_1 - x_0)^2.$$

$$U^{(1)} = (\frac{1}{\imath 2\pi\hbar})^{\frac{1}{2}} [det(-\frac{\partial}{\partial x_0} \frac{m(x_1 - x_0)}{t})]^{\frac{1}{2}} e^{\frac{\imath m}{\hbar 2t}(x_1 - x_0)^2} = \sqrt{\frac{1}{\imath 2\pi\hbar}} \sqrt{\frac{m}{t}} e^{\frac{\imath m}{\hbar 2t}(x_1 - x_0)^2} = \sqrt{\frac{m}{\imath 2\pi\hbar t}} e^{\frac{\imath m}{\hbar 2t}(x_1 - x_0)^2}$$

The second path :

$$x_{cl}^{(2)} = \begin{cases} x_0 - vt' & \text{for } 0 < t' < \frac{x_0}{v} \\ -x_0 + vt' & \text{for } \frac{x_0}{v} < t' < t \end{cases}$$

where

$$v = \frac{(x_1 + x_0)}{t}$$

$$\mathcal{A}_{cl}^{(2)} = \int_0^{\frac{x_0}{v}} \frac{1}{2} m\dot{x}_{cl,1}^{(2)} dt' + \int_{\frac{x_0}{v}}^t \frac{1}{2} m\dot{x}_{cl,2}^{(2)} dt' = \frac{m}{2} \frac{(x_1 + x_0)^2}{t^2} \frac{x_0}{x_1 + x_0} + \frac{m}{2} \frac{(x_1 + x_0)^2}{t^2} [t - \frac{x_0}{x_1 + x_0}] = \frac{m}{2} \frac{(x_1 + x_0)^2}{t}$$

$$-\frac{\partial^2 \mathcal{A}_{cl}^{(2)}}{\partial x_0 \partial x_1} = -\frac{\partial}{\partial x_0} \frac{m(x_1 + x_0)}{t} = -\frac{m}{t}$$

Hence :

$$U^{(2)} = (-1) * \sqrt{\frac{m}{\imath 2\pi\hbar t}} e^{\frac{\imath m}{\hbar 2t}(x_1+x_0)^2} = -\sqrt{\frac{m}{\imath 2\pi\hbar t}} e^{\frac{\imath m}{\hbar 2t}(x_1+x_0)^2}$$

And, finally, the propogator :

$$U(x|x_0) = \sqrt{\frac{m}{\imath 2\pi\hbar t}} [e^{\frac{\imath m}{\hbar 2t}(x_1-x_0)^2} - e^{\frac{\imath m}{\hbar 2t}(x_1+x_0)^2}]$$

What we can see from the solution, that actually we have two particles, coming from two different directions, without any allusion of the wall. Let us check the answer.At point $x = 0$, we have:

$$U(0|x_0) = \sqrt{\frac{m}{\imath 2\pi\hbar t}} [e^{\frac{\imath m}{\hbar 2t}(x_0)^2} - e^{\frac{\imath m}{\hbar 2t}(x_0)^2}] = 0$$

That means that there is **no** particle, which hit the wall, and this implied also by the Dirichlet boundary conditions; so the right ν is, as we described earlier, +2 and not +1.