E784: Optional expression for the total cross section

Submitted by: Noam Pinchas and Naor Yossi

The problem:

Show that the total cross section is approximately:

$$\sigma_{tot} = \frac{m^2}{\pi\left(\frac{h}{2\pi}\right)^4}\int_V\int_{V'} U(r)\,U(r')\,\frac{\sin^2(k|\vec{r}-\vec{r'}|)}{k^2|\vec{r}-\vec{r'}|^2}d^3r d^3r'$$

The solution:

$$\sigma_{tot} = \int_\Omega \frac{\partial\sigma}{\partial\Omega}d\Omega = \int_0^{2\pi}\int_0^{\pi}\frac{\partial\sigma}{\partial\Omega}\sin(\theta_\Omega)d\phi_\Omega d\theta_\Omega$$

After using the born approximation, we get:

$$\frac{\partial\sigma}{\partial\Omega} = \frac{1}{(2\pi\hbar)^2}\frac{{k_E}^2}{{v_E}^2}\left|\tilde{U}\left(\vec{k},\vec{k'}\right)\right|^2$$

$$\tilde{U}\left(\vec{k},\vec{k'}\right) = \int_{V'} U\left(\vec{r'}\right)\mathrm{e}^{i\left(\vec{k}-\vec{k'}\right)\vec{r'}}d^3r'$$

$$v_E = \frac{p}{m} \mapsto \frac{{k_E}^2}{{v_E}^2} = \frac{m^2}{\hbar^2}$$

$$k_E = \sqrt{2mE}\ ,\ k_E = k$$

$$\vec{k'} = k\hat{r}\ ,\ \vec{k} = k\hat{k}\ ,\ \hat{k}\hat{r} = \cos\theta_\Omega$$

$$\frac{\partial\sigma}{\partial\Omega} = \frac{m^2}{4\pi^2\hbar^4}\int_V U(\vec{r})\,\mathrm{e}^{i\left(\vec{k}-\vec{k'}\right)\vec{r}}d^3r\int_{V'} U\left(\vec{r'}\right)\mathrm{e}^{-i\left(\vec{k}-\vec{k'}\right)\vec{r'}}d^3r'$$

$$\frac{\partial\sigma}{\partial\Omega} = \frac{m^2}{4\pi^2\hbar^4}\int_V\int_{V'}\mathrm{e}^{i\vec{k}(\vec{r}-\vec{r'})}\mathrm{e}^{-i\vec{k'}(\vec{r}-\vec{r'})}U(\vec{r})\,U\left(\vec{r'}\right)d^3r d^3r'$$

$$\frac{\partial\sigma}{\partial\Omega} = \frac{m^2}{4\pi^2\hbar^4}\int_V\int_{V'}\mathrm{e}^{ik|\vec{r}-\vec{r'}|\cos\theta_\Omega}\mathrm{e}^{-ik'|\vec{r}-\vec{r'}|\cos\theta_{\Omega'}}U(\vec{r})\,U\left(\vec{r'}\right)d^3r d^3r'$$

$$\sigma_{tot} = \frac{m^2}{4\pi^2\hbar^4}\int_{\Omega}\int_{V}\int_{V'} \mathrm{e}^{ik|\vec{r}-\vec{r}'|\cos\theta_{\Omega}}\mathrm{e}^{-ik'|\vec{r}-\vec{r}'|\cos\theta_{\Omega'}}U\left(\vec{r}\right)U\left(\vec{r}'\right)\sin\left(\theta_{\Omega}\right)d\phi_{\Omega}d\theta_{\Omega}d^3rd^3r'$$

$$\sigma_{tot} = \frac{m^2}{4\pi^2\hbar^4}2\pi\int_{V}\int_{V'}\int_{-1}^{1} \mathrm{e}^{-ik|\vec{r}-\vec{r}'|\cos\theta_{\Omega'}}U\left(\vec{r}\right)U\left(\vec{r}'\right)\mathrm{e}^{ik|\vec{r}-\vec{r}'|\cos\theta_{\Omega}}d(\cos\theta_{\Omega})d^3rd^3r'$$

$$= \frac{m^2}{\pi\hbar^4}\int_{V}\int_{V'}\frac{\sin\left(k|\vec{r}-\vec{r}'|\right)}{k\left|\vec{r}-\vec{r}'\right|}U\left(\vec{r}\right)U\left(\vec{r}'\right)\mathrm{e}^{-ik|\vec{r}-\vec{r}'|\cos\theta_{\Omega'}}d^3rd^3r'$$

we now average over all incident beam direction $\vec{k}$ (assuming that V is spherically symmetric), than:

$$= \frac{m^2}{\pi\hbar^4}\int_{V}\int_{V'}\frac{\sin\left(k|\vec{r}-\vec{r}'|\right)}{k\left|\vec{r}-\vec{r}'\right|}U\left(r\right)U\left(r'\right)\left(\frac{1}{4\pi}\int_{\Omega'}\sin\left(\theta_{\Omega'}\right)d\phi_{\Omega'}d\theta_{\Omega'}\mathrm{e}^{-i\vec{k}(\vec{r}-\vec{r}')}\right)d^3rd^3r'$$

$$= \frac{m^2}{\pi\hbar^4}\int_{V}\int_{V'}\frac{\sin\left(k|\vec{r}-\vec{r}'|\right)}{k\left|\vec{r}-\vec{r}'\right|}U\left(r\right)U\left(r'\right)\left(\frac{1}{2}\int_{-1}^{1}\mathrm{e}^{-ik|\vec{r}-\vec{r}'|\cos\theta_{\Omega'}}d\left(\cos\theta_{\Omega'}\right)\right)d^3rd^3r'$$

$$= \frac{m^2}{\pi\left(\frac{h}{2\pi}\right)^4}\int_{V}\int_{V'}U\left(r\right)U\left(r'\right)\frac{\sin^2\left(k\left|\vec{r}-\vec{r}'\right|\right)}{k^2\left|\vec{r}-\vec{r}'\right|^2}d^3rd^3r'$$

As required, we derived:

$$\sigma_{tot} = \frac{m^2}{\pi\left(\frac{h}{2\pi}\right)^4}\int_{V}\int_{V'}U\left(r\right)U\left(r'\right)\frac{\sin^2\left(k\left|\vec{r}-\vec{r}'\right|\right)}{k^2\left|\vec{r}-\vec{r}'\right|^2}d^3rd^3r'$$