E7670: Gamow Decay to a ring

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The problem:

A particle of mass M is trapped inside a one dimensional device of length L_0. We will treat this device as a one dimensional potential well $(0 < x < L)$. The particle is prepared in one of the energy levels: $n = 1, 2, 3, 4, ...$

(1) Preparation question: Write the corresponding energy levels E_n and the eigenstates $\psi^n(x)$.

Definition: $v_n = (\frac{2E_n}{M})^{1/2}$

The particle can escape to a very long one-dimensional wire. Assume that the wire length is L (finite) in the middle stages of the solution. However, in the final stage this should be ignored.
The transmission coefficients of the connection points are: g_1, g_2.
Express your answers in terms of: (L_0, M, v_n, g)

(2) Assume that: $g_1 = g \ll 1, g_2 = 0$. Write an exact expression for the matrix elements V_{nk} between the device states and the wire states (suppose that: $E_k \approx E_n$).

(3) Assume that: $g_1 = g_2 = g \ll 1$. Write an exact expression for the density of the energy states $(\frac{dN}{dE})$ inside the long wire.

(4) Find the Gamow decay constant Γ_n for each state.

(5) Bonus question: explain how can a "lazy" student find the answer without any calculation.
Guidance and clues:
In order to solve this question one can use the following statement (no proof needed): when the two wires are being coupled weakly, one can treat the connection point as a "delta barrier": $V(r) = u\delta(r)$. The matrix elements between the states of one wire to the states of another are: $V_{nk} = -\frac{1}{4\mathsf{M}^2 u}[\partial\psi^{(n)}][\partial\Psi^{(k)}]$ where ∂ is the partial derivative in the radial direction (the coordinate r directed from the intersection outwards). The final answer should be expressed in terms of g. One can use the following statement (no proof needed): The traversablity of the delta barrier is very large: $g \approx (\frac{v_n}{u})^2$.

The solution:

(1) Since the device is treated as a one dimensional potential well $(0 < x < L)$, the particle states are of a particle in a one dimensional box. For a non-periodical boundary conditions $k_n = \frac{\pi n}{L_0}$
the energy levels are: $E_n = \frac{{k_n}^2}{2M} = \frac{\pi^2 n^2}{2ML_0}$
and the eigenstates are:

$$\psi^{(n)}(x) = \sqrt{\frac{2}{L_0}}\sin(k_n x) = \sqrt{\frac{2}{L_0}}\sin(\frac{\pi n x}{L_0})$$

for the long wire, substitute L_0 by L:

$$\psi^{(k)}(x) = \sqrt{\frac{2}{L}}\sin(k'_k x) = \sqrt{\frac{2}{L}}\sin(\frac{\pi k x}{L})$$

(2) The Hamiltonian is: $H = H_0 + V_{00}$
Let $\frac{\partial E}{\partial k} = \frac{k}{M} = v_E$ be the group velocity. $g = (\frac{v_n}{u})^2$ and the eigenstates of the device: E_n.
The difference between two momentum states are: $\Delta\Omega_d = \frac{\partial E}{\partial k}\Delta k = v_E \left(\frac{\pi(n+1)}{L_0} - \frac{\pi n}{L_0}\right) = v_E \frac{\pi}{L_0}$.
The eigenstates of the long wire: E_k and the difference between two momentum states is: $\Delta\Omega_w = v_E \frac{\pi}{L}$.

Now we find the matrix elements:

$$|V_{nk}| = -\frac{1}{4M^2 u}[\partial\psi^{(n)}][\partial\psi^{(k)}]$$

for the well wave function:

$$[\partial\psi^{(n)}] = \frac{\partial\psi}{\partial x}|_{x=0} = k_n \cos(x)|_{x=0} = k_n$$

similarly for the long wire wave function:

$$[\partial\psi^{(k)}] = k'_k$$

we get:

$$|V_{nk}| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}k_n k'_k$$

since: $E_k \approx E_n$, $v_k \approx v_n \approx v_E$, $k'_k \approx k_n \approx k_E$, $\sqrt{g} = \frac{v_E}{u}$, $\frac{k_E}{M} = v_E$
we get:

$$|V^{(1)}_{nk}(E)| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}(k_E)^2 = \frac{1}{4M^2}\frac{\sqrt{g}}{v_E}\frac{2}{\sqrt{LL_0}}v_E^2 M^2 = \frac{\sqrt{g}v_E}{2\sqrt{LL_0}}$$

(3) Now: $g1 = g2 = g$ therefore there is now coupeling in points: $(n,k) = (0,0)$ and $(n,k) = (k,0)$.
The Hamiltonian (in the position base) is: $H = H_0 + V_{00} + V_{L_0 0}$
The Hamiltonian (in the momentum base) is: $H = H_0 + V_{00} + V_{k0}$

$$|V^{(1)}_{nk}(0,0)| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}\cos(0)\cos(0)$$

$$|V^{(1)}_{nk}(L_0,0)| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}\cos(\pi n)\cos(0)$$

Thus, only the anti-symmetrical states are coupled:

$$|V^{(1+1)}_{n}| = |V^{(1)}_{nk} \pm V^{(1)}_{nk}| = \frac{\sqrt{g}v_E}{\sqrt{LL_0}}, n = even$$

$$|V^{(1+1)}_{n}| = |V^{(1)}_{nk} \pm V^{(1)}_{nk}| = 0, n = odd$$

The density of the energy states inside the long wire:

$$g(E) = \frac{1}{\Delta\Omega_w} = \frac{1}{v_E\frac{\pi}{L}} = \frac{L}{\pi v_E}$$

(4) The Gamow decay constant Γ_n:

$$\Gamma = \frac{2\pi}{\Delta}|V_{nk}|^2$$

$$\Delta = \frac{\partial E}{\partial k}\Delta k = v_E\left(\frac{\pi(n+1)}{L} - \frac{\pi n}{L}\right) = \frac{\pi v_E}{L}$$

therefore:

$$\Gamma_n = \frac{2\pi}{\frac{\pi v_E}{L}}\left(\frac{\sqrt{g}v_E}{\sqrt{LL_0}}\right)^2 = 2g\frac{v_n}{L_0}$$

(5) If we can treat the connection point as a "delta barrier": $V(r) = u\delta(x)$ and the connection point is: $x = 0$.

$$V_{nk} = -\frac{1}{4\mathsf{M}^2 u}[\partial\psi^{(n)}][\partial\Psi^{(k)}]$$

The first order pertubation exists only in: $[0, dx]$.

the wave functions:

$$\psi^{(n)}(x) = \sqrt{\frac{2}{L_0}}\sin(k_n x) = \sqrt{\frac{2}{L_0}}\sin(\frac{\pi n x}{L_0})$$

$$\psi^{(k)}(x) = \sqrt{\frac{2}{L}}\sin(k'_k x) = \sqrt{\frac{2}{L}}\sin(\frac{\pi k x}{L})$$

we get:

$$|V_{nk}| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}k_n k'_k$$

Since the traversablity of the delta barrier $g \approx (\frac{v_n}{u})^2$ is very large, substituting we get:

$$|V^{(1)}_{nk}(E)| = \frac{1}{4M^2 u}\frac{2}{\sqrt{LL_0}}(k_E)^2 = \frac{\sqrt{g}v_E}{2\sqrt{LL_0}}$$