E736: Oscillations in a two site system

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The problem:

A particle can is placed in a two site system. at $t = 0$ the particle is in the left site.
(1)find the probability $P(t)$ of the particle to stay in the left site.
the difference between the on site energies is ϵ, The hopping amplitude per unit time is Δ.
(2)compare the solution using first order pertrubation theory to the exact solution.(which could be easily found using the precession picture)

The solution:

(1)the Hamiltonian can be written as: $H = H_0 + Wf(t)$ where:

$$\mathcal{H}_\prime = \begin{pmatrix} \epsilon/2 & 0 \\ 0 & -\epsilon/2 \end{pmatrix}$$

$$W_{1,2} = W_{2,1} = \Delta$$

and

$$f(t') = 1 \quad 0 < t' < t$$

$$0 \quad otherwise$$

the probability to find the particle in a different site is given by:

$$P_t(n|n_0) = |W_{n,n_0}|^2 |FT(f(t')|^2$$

therefore using:

$$FT(f(t')) = \int_{-\infty}^{\infty} f(t')dt' = \frac{1 - e^{i(E_2 - E_1)t}}{E_2 - E_1} = -\frac{1 - e^{-i(\epsilon)t}}{i\epsilon}$$

we get the probabilty to find the particle in the right site:

$$P_t(2|1) = |W_{2,1}|^2 |FT(f(t'))|^2 = 2(\frac{\Delta}{\epsilon})^2 (1 - Cos(\epsilon t)) = 4(\frac{\Delta}{\epsilon})^2 Sin^2(\epsilon t/2)$$

and the probablity to find the particle in the left site is therefore:

$$P(t) = 1 - 4(\frac{\Delta}{\epsilon})^2 Sin^2(\epsilon t/2)$$

(2) the exact solution in the two state system can be found by solving the equations for $C_i(t)$:

$$i\frac{dc_1}{dt} = e^{i\epsilon t}\Delta C_2$$

$$i\frac{dc_2}{dt} = e^{-i\epsilon t}\Delta C_1$$

by extracting C_1 from the second equation and substituting in the first we errive to the equation:

$$\frac{d^2c_2}{dt^2} + i\epsilon\frac{dc_2}{dt} + \Delta^2 C_2 = 0$$

and by searching for a solution of the form:$C_2 = Ae^{\alpha t}$ the solution will be:

$$C_2 = e^{-i\epsilon t/2}[Ae^{i\sqrt{\epsilon^2+4\Delta^2}/2t} + Be^{-i\sqrt{\epsilon^2+4\Delta^2}/2t}]$$

using the initial conditions:$P_{t=0}(\text{particle in left site}) = 1$,$P_{t=0}(\text{particle in right site}) = 0$ the constants A and B are determined:

$$A = -B = \frac{2\Delta}{\sqrt{\epsilon^2 + 4\Delta^2}}$$

and finally:

$$P(t) = 1 - P_t(2|1) = 1 - 4(\frac{\Delta}{\sqrt{\epsilon^2 + 4\Delta^2}})^2 Sin^2(\sqrt{\epsilon^2 + 4\Delta^2}t/2)$$

and as we can see if we'd neglect the Δ under the root we'd get the same result in both cases