E6514: WKB Transmission of Harmonic and Morse Barriers

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The problem:

The exact "under the barrier" transmission coefficients for the repulsive 1D Harmonic ($-(1/2)Kx^2$) and Morse ($U_0/\cosh^2(\alpha x)$) potentials can be found in [Landau and Lifshitz][1]. Obtain the zero order WKB approximation coefficients for these potentials and compare them to the exact results.

The solution:

The transmission coefficient for a potential barrier entirely above the energy level under the zero order WKB approximation is given by,

$$T = \exp\left(-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2\mathsf{m}\left(V\left(x\right)-E\right)}dx\right)$$

The integration here occurs between the classical turning points. This approximation holds for sufficiently small values of T (That is, well below the peak in this problem) and sufficiently long wavelengths.

(1) The transmission coefficient for the 1D Repulsive Harmonic Potential given in [L.L] at §50 Problem 4, under the condition that $E_0/h \gg \sqrt{K/m}$ where $E_0 = |E|$, becomes

$$T = \frac{1}{1+\mathrm{e}^{2\pi(E_0/\hbar)\sqrt{\mathsf{m}/K}}} \approx \mathrm{e}^{-2\pi(E_0/\hbar)\sqrt{\mathsf{m}/K}}$$

The same result is obtained using WKB by evaluating the following integral between the two classical turning points ($x_{1,2} = \pm\sqrt{2E/K}$) using a $\sin\alpha = \sqrt{K/2E_0}x$ substitution,

$$\phi = \int_{x_1}^{x_2}\sqrt{2\mathsf{m}\left(E_0-\frac{1}{2}Kx^2\right)}dx = 2E_0\sqrt{\frac{\mathsf{m}}{K}}\int_0^{\pi}\sin^2\alpha d\alpha = \pi E_0\sqrt{\frac{\mathsf{m}}{K}}$$

and using the result in $T = \mathrm{e}^{-2\phi/\hbar}$.

(2) The transmission coefficient for the 1D Repulsive Morse Potential given in [L.L] at §25 Problem 4, under the condition that $U_0 \gg E \gg \hbar^2\alpha^2/2\mathsf{m}$, becomes

$$T = \left(1+\left(\frac{\cosh\left(\frac{\pi}{2}\sqrt{8\mathsf{m}U_0/\hbar^2\alpha^2-1}\right)}{\sinh\left(\pi\sqrt{2\mathsf{m}E}/\hbar\alpha\right)}\right)^2\right)^{-1} \approx -\mathrm{e}^{2\pi\left(\sqrt{2\mathsf{m}U_0}-\sqrt{2\mathsf{m}E}\right)/\hbar\alpha}$$

This result can also be obtained using WKB by evaluating the following integral between the two classical turning points (Which satisfy $C^2 \equiv U_0/E - 1 = \sinh^2(\alpha x)$) using a $t = \sinh(\alpha x)$ substitution leading to an unpleasant contour integral with a known result,

$$\phi = \int_{x_1}^{x_2}\sqrt{2\mathsf{m}\left(\frac{U_0}{\cosh^2\alpha x}-E\right)}dx = \frac{\sqrt{2\mathsf{m}E}}{\alpha}\int_{-C}^{C}\frac{\sqrt{C^2-t^2}}{1+t^2}dt = \frac{\sqrt{2\mathsf{m}E}}{\alpha}\pi\left(\sqrt{C^2+1}-1\right)$$

and using this result in $T = \mathrm{e}^{-2\phi/\hbar}$.

[1] L. D. Landau, E.M. Lifshitz "Quantum Mechanics: Non-Relativistic Theory. Vol. 3"