E6510: Finding energy levels using WKB

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The problem:

Find, using the WKB approximation, the energy levels of a one dimensional problem with a potential of $V(x) = sign(\alpha) \cdot x^{\alpha}, \quad x \geq 0$.

For what values of α is there a floor?

In particular, discuss the cases of $\alpha = 2, 0, -1$

The solution:

According to the WKB approximation: $\int_{x_1}^{x_2} P(x)\, dx = \pi\hbar\left(n + \frac{1}{2}\right)$. For x_1, x_2- the turning points in the energy E_n.

$$P(x) = \sqrt{2m\left[E_n - V(x)\right]} = \sqrt{2m\left[E_n - sign(\alpha) \cdot x^{\alpha}\right]}$$

We will notice that for $\alpha > 0$, $E_n > 0$. And for $\alpha < 0$, $E_n < 0$ (In this case $E_n > 0$ is allowed, but these are not bound states).
$x_1 = 0 \qquad x_2 = \left(sign(\alpha) \cdot E_n\right)^{1/\alpha}$

We will refer to $sign(\alpha)$ as S from now on.

$$\int_{x_1}^{x_2} P(x)dx = \sqrt{2m}\int_0^{(SE_n)^{1/\alpha}} \sqrt{E_n - S \cdot x^{\alpha}}dx = \sqrt{2m}\left(SE_n\right)^{\frac{1}{2}}\int_0^{(SE_n)^{1/\alpha}} \sqrt{S - \frac{x^{\alpha}}{E_n}}dx =$$
$$= \sqrt{2m}\left(SE_n\right)^{\frac{1}{2}}\int_0^{(SE_n)^{1/\alpha}} \sqrt{S - S\left(\frac{x}{(SE_n)^{1/\alpha}}\right)^{\alpha}}dx =$$
$$= \sqrt{2m}\left(SE_n\right)^{\frac{1}{2}+\frac{1}{\alpha}}\int_0^1 \sqrt{S\left(1 - t^{\alpha}\right)}dt$$

We will define the integral: $C(\alpha) = \int_0^1 \sqrt{S\left(1 - t^{\alpha}\right)}dt$ and we shall get:

$$\int_{x_1}^{x_2} P(x)dx = \sqrt{2m}(SE_n)^{\frac{1}{2}+\frac{1}{\alpha}}C(\alpha) = \pi\hbar\left(n + \frac{1}{2}\right)$$

Extracting E_n from the expression we get a solution for all $\alpha \neq 0$:

$$E_n = S\left(\frac{\pi\hbar\left(n + \frac{1}{2}\right)}{\sqrt{2m}C(\alpha)}\right)^{\frac{2\alpha}{\alpha+2}}$$

The condition for floor to exist is the convergence of the integral defining $C(\alpha)$.
The reason is when the integral converges, it means that until certain energy, there is a finite area in the phase domain, and therefore a finite number of allowed states, hence, there has to be a ground

state (floor). When the integral does not converge, there is infinite area in the phase domain, which means infinite number of states, hence, no ground state.

$$Integrand\left\{C\left(\alpha\right)\right\} = \sqrt{S\left(1-t^{\alpha}\right)} \approx \begin{cases} t^{\frac{\alpha}{2}} & \alpha<0 \\ 1 & \alpha>0 \end{cases} \text{ When } t\to 0.$$

Therefore, the integral always converges for $\alpha > 0$, but for $\alpha < 0$ the integral only converges for $\alpha > -2$.
Conclusion: There is a floor for each $\alpha > -2$.

<u>$\alpha = 2$, **Harmonic Oscilator**</u>
$C\left(2\right) = \frac{\pi}{4}$
$E_n = 2\sqrt{\frac{2}{m}}\hbar\left(n+\frac{1}{2}\right) = 2\hbar\omega\left(n+\frac{1}{2}\right)$
$\left\{1 = \frac{1}{2}m\omega^2 \Rrightarrow \omega = \sqrt{\frac{2}{m}}\right\}$
We got the expected result with an additional factor of 2, coming from the fact we only regarded the positive half of the x axis.

<u>$\alpha = -1$, **Hydrogen Atom**</u>
$C\left(-1\right) = \frac{\pi}{2}$
$E_n = -\left(\frac{\sqrt{2}\hbar\left(n+\frac{1}{2}\right)}{\sqrt{m}}\right)^{-2} = -\frac{m}{2\hbar^2\left(n+\frac{1}{2}\right)^2}$
We got the spectrum of the hydrogen atom, except the $\frac{1}{2}$ which was added to the denominator.

For the case of $\alpha = 0$ we will solve separately.
$V\left(x\right) = 0 \qquad P\left(x\right) = \sqrt{2mE_n}$
$\int_0^L P\left(x\right)dx = \int_0^L \sqrt{2mE_n}dx = L\sqrt{2mE_n} = \pi\hbar\left(n+\frac{1}{2}\right) \;\Rrightarrow\; E_n = \frac{\pi^2\hbar^2\left(n+\frac{1}{2}\right)^2}{2mL^2}$
We got the spectrum of an infinite potential well with soft boundary conditions.
$\triangle E_n = E_n - E_{n-1} = \frac{\pi^2\hbar^2}{2mL^2}\left[\left(n+\frac{1}{2}\right)^2 - \left(n-\frac{1}{2}\right)^2\right] = \frac{\pi^2\hbar^2 n}{mL^2}$
$\lim\limits_{L\to\infty} \triangle E_n = 0$
Meaning that in the limit of $L\to\infty$ the gap between neighboring energy levels is reduced to zero, and we get a continues spectrum.
Of course there is a floor since the energy has to be positive which fits the 'floor condition' from the general case.