E616: Corrections for hydrogen atom with a finite size proton

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The problem:

Assume the proton is a uniformly charged sphere of radius R.
Find the first order correction for the atoms energy levels.

The solution:

The potential left unchanged outside the nucleus :

$V(r) = -\frac{e^2}{r}$

While inside the sphere we can calculate the potential using gauss law :

$AE = 4\pi\rho V$

$4\pi r^2 E = \frac{16\pi^2}{3}\rho r^3$

We know that the charge density is :

$\rho = \frac{e}{\frac{4\pi}{3}R^3}$

Therfore :

$E = \frac{er}{R^3}$

Now we can intagrate in order to find the potential :

$V(r) = -e\int_r^R Rdr' - \frac{e^2}{R} = -\frac{e^2}{R^3}\int_r^R r'dr' - \frac{e^2}{R} = -\frac{e^2r^2}{2R}|_r^R - \frac{e^2}{R^3} = \frac{e^2r^2}{2R^3} - \frac{3e^2}{2R}$

So we see that :

$V(r) = -\frac{e^2}{r} \qquad r > R$

$V(r) = \frac{e^2r^2}{2R^3} - \frac{3e^2}{2R} \qquad r \leq R$

$H = H^0 + H'$

$H' = V_f - V_i$

$H' = 0 \qquad r > R$

$H' = \frac{e^2r^2}{2R^3} - \frac{3e^2}{2R} + \frac{e}{r} \qquad r \leq R$

$E^{[1]} =< nlm|H'|nlm >= \int_0^R R_{nl}^*(\frac{e^2r^2}{2R^3} - \frac{3e^2}{2R} + \frac{e}{r})R_{nl}r^2dr$

Therefore the first order correction for the ground state is :

$E_0^{[1]} =< 100|H'|100 >= \int_0^R \frac{e^{-\frac{2r}{a_0}}}{\pi a_0^3}(\frac{e^2r^2}{2R^3} - \frac{3e^2}{2R} + \frac{e}{r})r^2dr$

$E_0^{[1]} = \frac{e^2}{4\pi}(\frac{3a_0^3 - 3a_0R^2 + 3R^3 - 3a_0e^{\frac{-2R}{a_0}}(a_0+R)^2}{2R^3a_0})$