E492: An atom in a homogenic magnetic field (Zeeman effect)

Submitted by: Ofer Ginzburg

The problem:

The hamiltonian that describes the energy level E=E_{nl}

of an atom is H=const+vLS+$g_L L_z + g_S S_z$

(1) Explain the physical meaning of the prefactors.
(2) From now on consider the case where l=1.
(3) Write the hamiltonian in the standard basis.
(4) Show that one can find immediatly an exact solution for two eigenenergies.
(5) Show that in order to find the rest of the eigen-energies one needs two diagonalize two 2X2 matrices.
(6) What is the exact solution when $g_L = g_S$?

The solution:

Sections (1) through (4) were solved in the lecture notes, [26.3], pp. 108-109.

Attention: the equation describing the hamiltonian is slightly different in the notes. Notice the differences.

(5) The hamiltonian in the standard basis is (as found in the notes):

$$H = g_L \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} + g_S \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} + v \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

It's easy to see that the hamiltonian is block diagonal, with two 1X1 blocks at the 'edges' and two 2X2 blocks in the 'middle'. The 1X1 blocks give the two eigenenergies found in section (3): $E_{j=\frac{3}{2},m_j=\pm\frac{3}{2}} = \frac{v}{2} \pm (g_L + \frac{g_S}{2})$

In order to find the rest of the eigenenergies we need to diagonalize the 2X2 blocks, which form two different matrices:

$$\begin{pmatrix} g_L - \frac{g_S}{2} - \frac{v}{2} & \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} & \frac{g_S}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{g_S}{2} & \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} & -g_L + \frac{g_S}{2} - \frac{v}{2} \end{pmatrix}$$

(6) Let us denote $g_L = g_S = g$.

Then the matrices take the form:

$$\begin{pmatrix} \frac{g}{2} - \frac{v}{2} & \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} & \frac{g}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{g}{2} & \frac{v}{\sqrt{2}} \\ \frac{v}{\sqrt{2}} & -\frac{g}{2}-\frac{v}{2} \end{pmatrix}$$

The first matrix takes the eigenvalues (which are in fact the eigenenergies):

$\lambda_1 = \frac{g}{2} - v$

$\lambda_2 = \frac{g}{2} + \frac{v}{2}$

And from the second matrix we get:

$\lambda_1 = -\frac{g}{2} - v$

$\lambda_2 = -\frac{g}{2} + \frac{v}{2}$

To complete the six eigenenergies we bring the two eigenenergies found earlier:

$\mathrm{E}_1 = \frac{v}{2} + \frac{3}{2}g$

$\mathrm{E}_1 = \frac{v}{2} - \frac{3}{2}g$

Now the full diagonalized hamiltonian can be written as:

$$H = \begin{pmatrix} \frac{v}{2}+\frac{3}{2}g & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v}{2}-\frac{3}{2}g & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g}{2}-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g}{2}+\frac{v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{g}{2}-v & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{g}{2}+\frac{v}{2} \end{pmatrix}$$

Since $\mathrm{g}_L = g_S = g \quad -$

we can write the hamiltonian as follow:

H=const+vLS+gJ_z.

We know that the LS coupling relates directly to J^2.

And so to conclude, we know that the hamiltonian will commute with both $\mathrm{J}^2 \ and \ J_z$.