E489: Decay of a particle with spin 0

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The problem:

A particle at rest with spin 0 decays into 2 particles with spin 1/2 each. The Z component of the spin of each particle is measured. The detectors are placed with an angle θ with respect to the Z axis. Let us denote with $P(\theta)$ the probability of finding both particles with a state up, with S the total spin of the particles, and with L the mutual orbital angular momentum.

(1) What could be the orbital angular momentum of the particles after the decay?
(2) Write (schematically) the state of the system after the decay, in a basis set by L,S.
(3) Explain classically why $P(0) = 0$.
(4) Find an explicit expression for the ratio $P(\theta)/P(90°)$.

The solution:

`There is a second version of the solution in the pool`

(1) Before the decay, the particle had no spin and no angular momentum (it was at rest). Therefore, it is quite clear that

$$J^2\psi_0 = 0 \Longrightarrow j = 0 \tag{1}$$

Where ψ_0 denotes the state of the system. Due to angular momentum conservation, the new state of the system after the decay ψ must also fullfill

$$J^2\psi = 0 \Longrightarrow j = 0 \tag{2}$$

Let us denote by l the quantum number representing the orbital angular momentum of the system after the decay. Then by the addition of angular momentum rule,

$$\begin{aligned} [l] \otimes \left[\frac{1}{2}\right] \otimes \left[\frac{1}{2}\right] &= [l] \otimes ([1] \oplus [0]) \\ &= [l+1] \oplus [l] \oplus [|l-1|] \oplus [l] \end{aligned} \tag{3}$$

Combining eq.(2) and eq.(3) we find that the possible values for l are either 1 or 0.

(2) We denote by S the total spin of the particles, so that the quantum number s is

$$s = s_1 + s_2 = 1/2 + 1/2 = 1$$

And the quantum number m_s is

$$m_s = 1, 0, -1$$

Similarly, either

$$\begin{aligned} l &= 1 \\ m_l &= 1, 0, -1 \end{aligned}$$

Or

$$\begin{aligned} l &= 0 \\ m_l &= 0 \end{aligned}$$

Therefore, the dimension of the Hilbert (dropping radial dependence) is

$$dim = \underbrace{5 \oplus 3 \oplus 1 \oplus 3}_{l=1} \oplus \underbrace{3 \oplus 1}_{l=0} = 16 \tag{4}$$

But due to the postulates of the exercise (eq.(2)), we will find that ψ lives on a (much) smaller sub-space. Generally, we can define several bases to describe the Hilbert space:

$$\begin{aligned} spatially &\longrightarrow \{|\theta, \varphi; m_{s1}, m_{s2}\rangle\} & \text{(5a)}\\ L, S &\longrightarrow \{|l, s, m_l, m_s\rangle\} & \text{(5b)}\\ J &\longrightarrow \{|l, s, j, m_j\rangle\} & \text{(5c)} \end{aligned}$$

The basis (5c) is the best choice if we wish to determine the dimension of our sub-space, because by eq.(2) we find that the state of the system has to be

$$|\psi\rangle = A\,|l=0, s=1, j=0, m_j=0\rangle + B\,|l=1, s=1, j=0, m_j=0\rangle \tag{6}$$

Therefore, the dimension of our sub-space is 2, which corresponds to the fact that we found only two options to get $j=0$, either with $l=0$ or $l=1$. We wish to write the state of the system in the basis set by L,S (as the question requires):

$$\begin{aligned} |\psi\rangle &= \sum_{l,s;m_l=-l..l;m_s=-s..s} |l, s, m_l, m_s\rangle \langle l, s, m_l, m_s|\psi\rangle \\ &= A'\,|l=1, s=1, m_l=1, m_s=-1\rangle + B'\,|l=1, s=1, m_l=-1, m_s=1\rangle \\ &+ C'\,|l=1, s=1, m_l=0, m_s=0\rangle + D'\,|l=0, s=1, m_l=0, m_s=0\rangle \end{aligned} \tag{7}$$

Where

$$\begin{aligned} A' &= \langle l=1, s=1, m_l=1, m_s=-1|\psi\rangle \\ &= B\,\langle l=1, s=1, m_l=1, m_s=-1|l=1, s=1, j=0, m_j=0\rangle & \text{(8a)}\\ B' &= \langle l=1, s=1, m_l=-1, m_s=1|\psi\rangle \\ &= B\,\langle l=1, s=1, m_l=-1, m_s=1|l=1, s=1, j=0, m_j=0\rangle & \text{(8b)}\\ C' &= \langle l=1, s=1, m_l=0, m_s=0|\psi\rangle \\ &= B\,\langle l=1, s=1, m_l=0, m_s=0|l=1, s=1, j=0, m_j=0\rangle & \text{(8c)}\\ D' &= \langle l=0, s=1, m_l=0, m_s=0|\psi\rangle \\ &= A\,\langle l=0, s=1, m_l=0, m_s=0|l=0, s=1, j=0, m_j=0\rangle & \text{(8d)} \end{aligned}$$

Solving eqs.(8) is difficult, because it would require us to compute the transition matrix, using the operators J_x, J_y which "live" in the full Hilbert space, whose dimension is given in eq.(4). However, it is easy to see from eqs.(8) that the coefficients depend only on 2 free parameters, the same A and B as before.

(3) I'm afraid the answer still eludes me. I have discussed the question with DC, but after further thought I came to disagree with his point of view, which I will lay out here, to promote discussion on the matter.
In order for measurement to be taken, the particles have to reach the detectors, so when we say that the detector is placed at $\theta = 0$ it means that one measured particle has moved up while the other has moved down. Each particle has in that case a velocity component only on the Z axis, and so the orbital angular momentum is therefore 0. Satisfaction of eq.(2) requires that the particles have opposite spins.
I do not subscribe to that explanation, because that explanation relies on that the particles are moving on a straight line, but if that was true, then there would not be an orbital angular momentum

(classically), no matter the angle which we put the detectors at. In order for us to get a finite probability, the particles have to perform a complex motion in space, not on a linear line, so that there would be a finite orbital momentum, which would allow both the spins to be in the "up" direction (with respect to the Z axis).

(4) The probability $P(\theta)$ is best defined using the spatial basis ((5)a), where both spins are in the "up" state, i.e.

$$m_{s1} = +1/2,\ m_{s2} = +1/2 \Longrightarrow m_s = +1 \tag{9}$$

Using eq.(7), it is easy to see that the probability is governed by the following projection:

$$\langle \theta, \varphi, m_{s1} = +1/2, m_{s2} = +1/2|\psi\rangle = B' \langle \theta, \varphi, m_{s1} = +1/2, m_{s2} = +1/2|l = 1, s = 1, m_l = -1, m_s = 1\rangle \tag{10}$$

That is, as far as the probability $P(\theta)$ is concerned, the only part of the state's wave function (eq.(7)) which matters is $|l = 1, s = 1, m_l = -1, m_s = 1\rangle$. Assuming no magnetic field is present, and that the detectors are placed far enough so that spin-orbit interaction can be neglected, then

$$\langle r, \theta, \varphi|l = 1, s = 1, m_l = -1, m_s = 1\rangle \sim Y_{l=1}^{m_l=-1}(\theta, \varphi) \tag{11}$$

Since

$$Y_{l=1}^{m_l=-1}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} sin(\theta)\, \mathrm{e}^{-i\varphi} \tag{12}$$

It is easy to see that

$$P(\theta)/P(90°) = sin^2(\theta) \tag{13}$$