E487: Spin $\frac{1}{2}$+Spin $\frac{1}{2}$+Spin $\frac{1}{2}$

Submitted by: Ronit Jan Kletter

The problem:

(1) Decompose the standard representation of Spin $\frac{1}{2}$+Spin $\frac{1}{2}$+Spin $\frac{1}{2}$
(2) Write the transition matrix T from the old base to the new one

The solution:

`There is a second version of the solution in the pool`

(1)

$$\begin{aligned}
\left[\frac{1}{2}\right] \otimes \left[\frac{1}{2}\right] \otimes \left[\frac{1}{2}\right] &= \left[\frac{1}{2}\right] \otimes ([1] \oplus [0]) \\
&= \left[\frac{1}{2}\right] \otimes [1] \oplus \left[\frac{1}{2}\right] \otimes [0] \\
&= \left[\frac{3}{2}\right] \oplus \left[\frac{1}{2}\right] \oplus \left[\frac{1}{2}\right] \\
&= 4 \oplus 2 \oplus 2
\end{aligned}$$

(2)

The standard representation of Spin $\frac{1}{2}$+Spin $\frac{1}{2}$+Spin $\frac{1}{2}$ gives us 8 states as follows:

$|\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\downarrow\rangle$

Let us define a decending operator of the form:

$J_- = L_- + S_- + N_-$

Where

$L_- |m_l, m_s, m_n\rangle = \sqrt{l(l+1) - m_l(m_l-1)}\, |m_l - 1, m_s, m_n\rangle$

$S_- |m_l, m_s, m_n\rangle = \sqrt{s(s+1) - m_s(m_s-1)}\, |m_l, m_s - 1, m_n\rangle$

$N_- |m_l, m_s, m_n\rangle = \sqrt{n(n+1) - m_n(m_n-1)}\, |m_l, m_s, m_n - 1\rangle$

We operate on the higher state : $|\uparrow\uparrow\uparrow\rangle$

$J_- |\uparrow\uparrow\uparrow\rangle = L_- |\uparrow\uparrow\uparrow\rangle + S_- |\uparrow\uparrow\uparrow\rangle + N_- |\uparrow\uparrow\uparrow\rangle$

Using the definitions of the L_-, S_-, N_-, we get the second level state:

$J_- |\uparrow\uparrow\uparrow\rangle = \frac{1}{\sqrt{3}} |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle$

In order to get the third level state, we use the decending operator again

$J_- \frac{1}{\sqrt{3}} (|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) = \frac{1}{\sqrt{3}} (|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle)$

The forth level state is simply $|\downarrow\downarrow\downarrow\rangle$

The second and third levels are degenerate and we need to find the orthogonal states, two states for each level.

For the second level:

The first state is simply $0 \cdot |{\downarrow\uparrow\uparrow}\rangle + 1 \cdot |{\uparrow\downarrow\uparrow}\rangle - 1 \cdot |{\uparrow\uparrow\downarrow}\rangle \Rightarrow \frac{1}{\sqrt{2}} \left(|{\uparrow\downarrow\uparrow}\rangle - |{\uparrow\uparrow\downarrow}\rangle\right)$

The second state has to be orthogonal to both states with the same j value and therefore equals to:

$-2 \cdot |{\downarrow\uparrow\uparrow}\rangle + 1 \cdot |{\uparrow\downarrow\uparrow}\rangle + 1 \cdot |{\uparrow\uparrow\downarrow}\rangle \Rightarrow \frac{1}{\sqrt{6}} \left(-2 \cdot |{\downarrow\uparrow\uparrow}\rangle + 1 \cdot |{\uparrow\downarrow\uparrow}\rangle + 1 \cdot |{\uparrow\uparrow\downarrow}\rangle\right)$

For the third level:

By operating J_- on the first and second orthogonal states respectively, we get an additional dublet

$J_- \left(|{\uparrow\downarrow\uparrow}\rangle - |{\uparrow\uparrow\downarrow}\rangle\right) = |{\downarrow\downarrow\uparrow}\rangle + |{\uparrow\downarrow\downarrow}\rangle - |{\downarrow\uparrow\downarrow}\rangle - |{\uparrow\downarrow\downarrow}\rangle = \frac{1}{\sqrt{2}} \left(|{\downarrow\downarrow\uparrow}\rangle - |{\downarrow\uparrow\downarrow}\rangle\right)$

$J_- \left(-2 \cdot |{\downarrow\uparrow\uparrow}\rangle + 1 \cdot |{\uparrow\downarrow\uparrow}\rangle + 1 \cdot |{\uparrow\uparrow\downarrow}\rangle\right) \Rightarrow \frac{1}{\sqrt{6}} \left(-|{\downarrow\downarrow\uparrow}\rangle - |{\downarrow\uparrow\downarrow}\rangle + 2\,|{\uparrow\downarrow\downarrow}\rangle\right)$

In order to construct the transition matrix, let us mark the states:

$|1\rangle = |{\uparrow\uparrow\uparrow}\rangle$
$|2\rangle = |{\uparrow\uparrow\downarrow}\rangle$
$|3\rangle = |{\uparrow\downarrow\uparrow}\rangle$
$|4\rangle = |{\downarrow\uparrow\uparrow}\rangle$
$|5\rangle = |{\downarrow\uparrow\downarrow}\rangle$
$|6\rangle = |{\downarrow\downarrow\uparrow}\rangle$
$|7\rangle = |{\uparrow\downarrow\downarrow}\rangle$
$|8\rangle = |{\downarrow\downarrow\downarrow}\rangle$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-2}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$