ex:4860 Angular momentum considerations in a particle decay

Submitted by: Amir Fruchtman

The problem:
1. write down in the standard base the states $|j, m = 0\rangle$ of a system that has two spin 1 particles

2. a particle with spin 0 in the free space decays into two stationary particles with spin 1. their linear polarization is measured. what is the probability that they are both polarized in the same direction?

in the following questions the wave function is determined uniquely by three considerations:
fermions / bosons ; reflection symmetry; the decay interaction at the origin is point-like.

3. a particle with spin 0 is forces to decay into two particles with spin 1/2, so one of them moves up with momentum k and the other down with reversed momentum. write down in the standard base the quantum state of the two spins.

4. a point-like particle with spin 0 is forced to decay into two particles with spin 1 so one is moving up the the other is moving down. suppose both the spins don't have linear polarization in their moving direction, and write down in the standard base the quantum state of the two spins.

5. in the last case, what is the probability that the total spin j of the outcome is zero?
note: because this problem has a preferred direction, the total spin is not a constant of motion, therefore does not have to be zero.

the answer:
1. In this problem we need to find the Clebsch Gordan coefficients for the case of $3 \otimes 3 = 5 \oplus 3 \oplus 1$.
in this case the standard base is

$$|\Uparrow\Uparrow\rangle, |\Updownarrow\Updownarrow\rangle, |\Downarrow\Downarrow\rangle, |\Uparrow\Updownarrow\rangle, |\Updownarrow\Uparrow\rangle, |\Updownarrow\Downarrow\rangle, |\Downarrow\Updownarrow\rangle, |\Uparrow\Downarrow\rangle, |\Downarrow\Uparrow\rangle.$$

so let't start from the state $|2,2\rangle = |\Uparrow\Uparrow\rangle$:

$$|2,1\rangle \propto J_-|\Uparrow\Uparrow\rangle = |\Uparrow\Updownarrow\rangle + |\Updownarrow\Uparrow\rangle$$

$$|2,0\rangle \propto J_-J_-|\Uparrow\Uparrow\rangle = |\Uparrow\Downarrow\rangle + |\Updownarrow\Updownarrow\rangle + |\Updownarrow\Updownarrow\rangle + |\Downarrow\Uparrow\rangle = |\Uparrow\Downarrow\rangle + 2|\Updownarrow\Updownarrow\rangle + |\Downarrow\Uparrow\rangle$$

the $|1,1\rangle$ state is given by orthogonality to state $|2,1\rangle$:

$$|1,1\rangle \propto |\Uparrow\Updownarrow\rangle - |\Updownarrow\Uparrow\rangle$$

so

$$|1,0\rangle \propto J_-(|\Uparrow\Updownarrow\rangle - |\Updownarrow\Uparrow\rangle) = |\Uparrow\Downarrow\rangle + |\Updownarrow\Updownarrow\rangle - |\Updownarrow\Updownarrow\rangle - |\Downarrow\Uparrow\rangle = |\Uparrow\Downarrow\rangle - |\Downarrow\Uparrow\rangle$$

again, by orthogonality to states $|2,0\rangle$ and $|1,0\rangle$ we can get the state:

$$|0,0\rangle \propto |\Uparrow\Downarrow\rangle - |\Updownarrow\Updownarrow\rangle + |\Downarrow\Uparrow\rangle$$

now let us normalize:

$$|2,0\rangle = \frac{1}{\sqrt{6}}(|\Uparrow\Downarrow\rangle + 2|\Updownarrow\Updownarrow\rangle + |\Downarrow\Uparrow\rangle)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\Uparrow\Downarrow\rangle - |\Downarrow\Uparrow\rangle)$$

$$|0,0\rangle = \frac{1}{\sqrt{3}}(|\Uparrow\Downarrow\rangle - |\Updownarrow\Updownarrow\rangle + |\Downarrow\Uparrow\rangle)$$

2. because of angular-momentum conservation, the resulting particles must have $j = 0$ therefore they are in the $|0,0\rangle$ state. a linear polarization of a spin in the $\hat{z}$ direction is denoted by $\Updownarrow$.
so it is obvious that the probability of both spins to be polarized in the $\hat{z}$ direction is

$$P_{zz} = |\langle\Updownarrow\Updownarrow|0,0\rangle|^2 = 1/3$$

since the problem is isotropic, the probability of having both spins both be polarized in $\hat{x}$ or $\hat{y}$ direction is 1/3, hence the answer is multiplied by 3:

$$P = 1 = P_{xx} + P_{yy} + P_{zz}$$

which means the spins have to be polarized in the same direction.

3. spin 1/2 particles are fermions so they must obey the Fermi statistics, hence they have to be in the singlet:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

4. spin 1 particles are bosons so they must obey the Bose-Einstein statistics. therefore they must be a linear combination of the symmetric states $|2,0\rangle, |0,0\rangle$. since we cant have linear polarization we must get rid of the $|\Updownarrow\Updownarrow\rangle$ term, leaving us with the only option:

$$\frac{1}{\sqrt{2}}(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle)$$

5. the probability is calculated by:

$$P = |\langle 0,0|\frac{1}{\sqrt{2}}(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle)|^2 = \frac{1}{6}|(\langle\Uparrow\Downarrow| - \langle\Updownarrow\Updownarrow| + \langle\Downarrow\Uparrow|)(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle)|^2$$

and is therefore

$$P = 2/3$$