Ex4857: Dynamics of Coupled Spins

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The problem:

There are 2 particles with spin 1/2. The system is described via the Hamiltonian:

$$\mathcal{H} = \mu S^A \cdot S^B - h \cdot \left(S^A + S^B\right)$$

Where the magnetic field is in Z axis.
The states of the system are defined in the **standard basis**: $|ZZ\rangle, |Z\bar{Z}\rangle, |\bar{Z}Z\rangle, |\bar{Z}\bar{Z}\rangle$

In experiment I the system was prepared in state $|XX\rangle$.
In experiment II the system was prepared in state $|ZX\rangle$.

In this question you are required to calculate the polarization vector $\vec{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ Which describes the polarization of particle B.
When answering (1), use the standard basis described above.

(1) Write the Matrix representation of the operator S_x which describes the spin of particle B.
(2) Find the polarization vector after a duration t in experiment I.
(3) Find the polarization vector after a duration t in experiment II.

The solution:

(1) We denote particle B as the one with ”fast counter”. Then we get by outer multiplication:

$$S_j^B = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$$

Specifically:

$$S_x^B = \frac{1}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

(2) We can write the Hamiltonian as follows, while remembering counting convention:

$$\mathcal{H} = \frac{\mu}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - h\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We notice that in the subspace of $|Z\bar{Z}\rangle, |\bar{Z}Z\rangle$ the Hamiltonian decompose into $-1 + 2\sigma_x$. Choosing the symmetric and anti-symmetric basis in this subspace yields the singlet-triplet basis for the Hamiltonian:

$$(|w_1\rangle, |w_2\rangle, |w_3\rangle, |w_4\rangle) = \left(|ZZ\rangle, \frac{1}{\sqrt{2}}\left(|Z\bar{Z}\rangle + |\bar{Z}Z\rangle\right), \frac{1}{\sqrt{2}}\left(|Z\bar{Z}\rangle - |\bar{Z}Z\rangle\right), |\bar{Z}\bar{Z}\rangle\right)$$

Which is expected because $2 \otimes 2 = 3 \oplus 1$.

So in this basis the Hamiltonian takes the form:

$$\mathcal{H} = \begin{pmatrix} -h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & h \end{pmatrix} + \text{Const.}$$

In the case of $|\psi_0\rangle = |XX\rangle = \frac{1}{2}\left(|w_1\rangle + |w_4\rangle + \sqrt{2}|w_2\rangle\right)$ we are contained in the triplet subspace. It is clear that the effective Hamiltonian is $-hS_z$ of spin 1. Therefore, the evolution operator is:

$$U = \mathrm{e}^{-i\mathcal{H}t} = \mathrm{e}^{-i\vec{\Omega}t\cdot\vec{S}}$$

Where $\vec{\Omega} = (0, 0, -h)$. Therefore the second spin precess clockwise around $\hat{z}$ with frequency h. As a result, the polarization vector is:

$$\vec{M}(t) = (cos(ht), -sin(ht), 0)$$

(3) Now the initial state is: $|\psi_0\rangle = |ZX\rangle = \frac{1}{2}\left(\sqrt{2}|w_1\rangle + |w_2\rangle + |w_3\rangle\right)$.

Expressed in the singlet-triplet basis, the time evolution of this state is:

$$|\psi_t\rangle = \frac{1}{2}\left(\sqrt{2}|w_1\rangle \mathrm{e}^{iht} + |w_2\rangle + |w_3\rangle \mathrm{e}^{i\mu t}\right)$$

Using the definition of $\vec{M}$ we calculate the observables: $\langle S_j\rangle = \sum_k \langle\psi_t|w_k\rangle\langle w_k|S_j|w_k\rangle\langle w_k|\psi_t\rangle$.

The spin matrices in the new basis are:

$$\tilde{S}_x = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\tilde{S}_y = \frac{i}{2\sqrt{2}}\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\tilde{S}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

As a result the polarization vector is:

$$\vec{M} = \frac{1}{2}\begin{bmatrix} cos\left(\frac{\mu t}{2}\right) cos\left(\left(\frac{\mu}{2} - h\right)t\right) \\ cos\left(\frac{\mu t}{2}\right) sin\left(\left(\frac{\mu}{2} - h\right)t\right) \\ sin^2\left(\frac{\mu t}{2}\right) \end{bmatrix}$$