Ex4856: Dynamics of coupled spins

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The problem:
Two coupled spins are described by the Hamiltonian:

$$\mathcal{H} = \mu S^A \cdot S^B - \overrightarrow{h} \cdot (S^A + S^B)$$

The magnetic field h is pointed towards the Z axis.
The spin states of the system are represented in the standard basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

In experiment (a) the system was prepared in state $|\Psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$.
In experiment (b) the system was prepared in state $|\Psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$.

In this question you need to calculate the polarization vector $\overrightarrow{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ which describes the polarization state of the second spin.
In your answer for section (1) you should use the standard basis.
The answers for sections (4,5) do not require hard work if you work wisely.

(1) Write down the matrix representation of the S_x operator which describes the spin of the second particle.

(2) Find the polarization vector which describes the initial polarization state of the second particle in experiment (a).

(3) Find the polarization vector which describes the initial polarization state of the second particle in experiment (b).

(4) What is the answer for section (2) after time t ?

(5) What is the answer for section (3) after time t ?

The solution:

(1) The S_x operator of the second spin is represnted as:

$$S_x^B = \hat{I}_{2\times 2} \otimes S_x = \hat{I}_{2\times 2} \otimes \frac{\sigma_x}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(2) The S_y, S_z operators of the second spin are represnted as:

$$S_y^B = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$S_z^B = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

After normalization, the wave function at time $t = 0$ as represented in the standard basis is:

$$|\Psi_a\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The polarization vector components are calculated by:

$$\langle S_x \rangle = \langle \Psi_a | S_x | \Psi_a \rangle = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$$\langle S_y \rangle = \langle \Psi_a | S_y | \Psi_a \rangle = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle S_z \rangle = \langle \Psi_a | S_z | \Psi_a \rangle = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6}$$

So the polarization vector:

$$\overrightarrow{M_a} = \left(\langle S_x^B \rangle, \langle S_y^B \rangle, \langle S_z^B \rangle \right) = \left(\frac{1}{3}, 0, \frac{1}{6} \right)$$

(3) Here the wave function at time $t = 0$ is

$$|\Psi_b\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

The same calculation as in section (2) yields the same result:

$$\overrightarrow{M_b} = \left(\langle S_x^B \rangle, \langle S_y^B \rangle, \langle S_z^B \rangle \right) = \left(\frac{1}{3}, 0, \frac{1}{6} \right)$$

(4) Let us divide the Hamiltonian into two components: $\mathcal{H} = \mathcal{H}_\mu + \mathcal{H}_h$.

$$\mathcal{H}_\mu = \mu S^A \cdot S^B = \frac{\mu}{2} \left(\left(S^A + S^B \right)^2 - S^{A2} - S^{B2} \right)$$

using the singlet-triplet basis we note that $\mathcal{H}_\mu$ is diagonal and that:

$$\mathcal{H}_\mu^{[l]} = \frac{\mu}{2} \left(l\,(l+1) - \frac{1}{2} \left(\frac{1}{2} + 1 \right) - \frac{1}{2} \left(\frac{1}{2} + 1 \right) \right) = \mu \begin{pmatrix} -\frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Since $h = h\hat{z}$ we get:

$$\mathcal{H}_h = -h(S_z^A + S_z^B) = h \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Presented in the singlet-triplet basis, ordered $|0,0\rangle, |1,-1\rangle, |1,0\rangle, |1,1\rangle$.
The total Hamiltonian, after gauge:

$$\mathcal{H} = \begin{pmatrix} -\mu & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h \end{pmatrix}$$

We note that $|\Psi_a\rangle$ is composed only from triplet states. The effective Hamiltonian in the triplet subspace is:

$$\mathcal{H}_h = -h(S_z^A + S_z^B) = \vec{\Omega} \cdot \vec{S}$$

Where $\vec{\Omega} = (0, 0, -h)$.
The time evolution of the wave function, and hence the polarization vector, is in fact precession around the $\hat{z}$ axis:

$$\vec{M}(t) = R_z(-ht)\vec{M}(0) = \frac{1}{3}\begin{pmatrix}\cos ht, & -\sin ht, & \frac{1}{2}\end{pmatrix}$$

(5) Now, as the singlet state is also present in the wave function, we will resort to performing some calculations:

$$|\Psi_b(0)\rangle = \frac{1}{\sqrt{3}}\left(|11\rangle + \sqrt{2}\,|00\rangle\right)$$

After time evolution:

$$|\Psi_b(t)\rangle = \frac{1}{\sqrt{3}}\left(|11\rangle\, \mathrm{e}^{iht} + \sqrt{2}\,|00\rangle\, \mathrm{e}^{i\mu t}\right)$$

So back in the standard basis:

$$|\Psi_b(t)\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\rangle\, \mathrm{e}^{iht} + |\uparrow\downarrow\rangle\, \mathrm{e}^{i\mu t} - |\downarrow\uparrow\rangle\, \mathrm{e}^{i\mu t}\right)$$

Calculating the polarization vectors:

$$\langle S_x\rangle = \frac{1}{6}\begin{pmatrix}\mathrm{e}^{-iht} & \mathrm{e}^{-i\mu t} & -\mathrm{e}^{-i\mu t} & 0\end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{iht} \\ \mathrm{e}^{i\mu t} \\ -\mathrm{e}^{i\mu t} \\ 0 \end{pmatrix} = \frac{1}{3}\cos(\mu - h)\,t$$

$$\langle S_y\rangle = \frac{1}{6}\begin{pmatrix}\mathrm{e}^{-iht} & \mathrm{e}^{-i\mu t} & -\mathrm{e}^{-i\mu t} & 0\end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{iht} \\ \mathrm{e}^{i\mu t} \\ -\mathrm{e}^{i\mu t} \\ 0 \end{pmatrix} = \frac{1}{3}\sin(\mu - h)\,t$$

$$\langle S_z \rangle = \frac{1}{6}\begin{pmatrix} \mathrm{e}^{-iht} & \mathrm{e}^{-i\mu t} & -\mathrm{e}^{-i\mu t} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{iht} \\ \mathrm{e}^{i\mu t} \\ -\mathrm{e}^{i\mu t} \\ 0 \end{pmatrix} = \frac{1}{6}$$

$$\overrightarrow{M}_b(t) = \frac{1}{3}\begin{pmatrix} \cos(\mu - h)\,t, & \sin(\mu - h)\,t, & \frac{1}{2} \end{pmatrix}$$