Ex4855: Dynamics of coupled spins

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The problem:

Two particles with spin 1/2. The interaction between the two spins is $\mu\vec{S}^A \cdot \vec{S}^B$. The spin states of the system are represented in the standard basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle$.

In experiment α the system was prepared in state $|\psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$.
In experiment β the system was prepared in state $|\psi\rangle \propto |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$.
In this question you need to calculate the polarization vector $\overrightarrow{M} = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle)$ which describe the initial polarization of the second spin.

In your answer for section (1) you should use the standard base.
The answers for sections (4,5) are not requiring "black work" if you work smart.

(1) Write down the matrix presentation of S_x operator that describes the spin of the second particle.

(2) Find the polarization vector that describes the initial polarization of the second particle in experiment α.

(3) Find the polarization vector that describes the initial polarization of the second particle in experiment β.

(4) What is the answer for section (2) after time t?

(5) What is the answer for section (3) after time t?

The solution:

(1) We can define S_x operator that describes the spin of the second particle as:

$$S_x^B = \hat{I}_{2\times2} \otimes S_x = \hat{I}_{2\times2} \otimes \frac{\sigma_x}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(2) In the same way we can calculate S_y^B, S_z^B:

$$S_y^B = \hat{I}_{2\times2} \otimes S_y = \frac{i}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_z^B = \hat{I}_{2\times2} \otimes S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The normalized wave function at time t=0, which prepared in experiment α is:

$$|\psi_\alpha\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

So the polarization vector components are:

$$\langle S_x\rangle = \langle\psi_\alpha|S_x^B|\psi_\alpha\rangle = \frac{1}{6}\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \frac{1}{3}$$

$$\langle S_y\rangle = \langle\psi_\alpha|S_y^B|\psi_\alpha\rangle = \frac{i}{6}\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & -1\\0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = 0$$

$$\langle S_z\rangle = \langle\psi_\alpha|S_z^B|\psi_\alpha\rangle = \frac{1}{6}\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \frac{1}{6}$$

And the polarization vector:

$$\overrightarrow{M_\alpha} = (\langle S_x^B\rangle, \langle S_y^B\rangle, \langle S_z^B\rangle) = (\frac{1}{3}, 0, \frac{1}{6})$$

(3) The normalized wave function at time t=0, which prepared in experiment β is:

$$|\psi_\beta\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}$$

And the polarization vector components are:

$$\langle S_x\rangle = \langle\psi_\beta|S_x^B|\psi_\beta\rangle = \frac{1}{3}$$

$$\langle S_y\rangle = \langle\psi_\beta|S_y^B|\psi_\beta\rangle = 0$$

$$\langle S_z\rangle = \langle\psi_\beta|S_z^B|\psi_\beta\rangle = \frac{1}{6}$$

So the polarization vector is:

$$\overrightarrow{M_\beta} = (\frac{1}{3}, 0, \frac{1}{6})$$

(4) We can write down the hamiltonian in the standard basis:

$$H = \mu(S_x^A S_x^B + S_y^A S_y^B + S_z^A S_z^B) = \frac{\mu}{4}\left[\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right] =$$

$$\frac{\mu}{4}\left[\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}+\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\right]=\frac{\mu}{4}\begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$$

Since the states in the standard basis are not the eigenstates of the Hamiltonian, we prefer to transfer the Hamiltonian to the Singlet-Triplet basis:

$$H|0,0\rangle = H[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)] = -\frac{3}{4}\mu|0,0\rangle$$

$$H|1,1\rangle = H|\uparrow\uparrow\rangle = \frac{\mu}{4}|1,1\rangle$$

$$H|1,0\rangle = H[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)] = \frac{\mu}{4}|1,0\rangle$$

$$H|1,-1\rangle = H|\downarrow\downarrow\rangle = \frac{\mu}{4}|1,-1\rangle$$

So eventually the Hamiltonian can be written in the new basis:

$$\widetilde{H} = \begin{pmatrix}\frac{\mu}{4}&0&0&0\\0&\frac{\mu}{4}&0&0\\0&0&\frac{\mu}{4}&0\\0&0&0&-\frac{3}{4}\mu\end{pmatrix}$$

By exploiting the freedom of adding a constant to the Hamiltonian, we can define $constant = -\frac{\mu}{4}$ and by adding it to the Hamiltonian we get:

$$\widetilde{H} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-\mu\end{pmatrix}$$

So Triplet states are constant in time and Singlet state change in time with eigenenergy μ. Now, we may write the evolution of the wavefunction in time:

$$|\psi_\alpha\rangle = \frac{1}{\sqrt{3}}(|11\rangle + \sqrt{2}|10\rangle) \Rightarrow |\psi_\alpha(t)\rangle = \frac{1}{\sqrt{3}}(|11\rangle + \sqrt{2}|10\rangle) = |\psi_\alpha\rangle$$

The wavefunction is consisted only from Triplet states, so it is constant in time. Therefore the polarization vector is constant in time as well:

$$\overrightarrow{M_\alpha}(t) = \overrightarrow{M_\alpha} = (\frac{1}{3}, 0, \frac{1}{6})$$

(5) The evolution of the wavefunction in time in experimet β:

$$|\psi_\beta\rangle = \frac{1}{\sqrt{3}}(|11\rangle + \sqrt{2}|00\rangle) \Rightarrow |\psi_\beta(t)\rangle = \frac{1}{\sqrt{3}}(|11\rangle + \sqrt{2}e^{i\mu t}|00\rangle) = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\e^{i\mu t}\\-e^{i\mu t}\\0\end{pmatrix}$$

So the polarization vector in time t will be:

$$\langle S_x\rangle = \langle\psi_\beta(t)|S_x^B|\psi_\beta(t)\rangle = \frac{1}{6}\begin{pmatrix}1 & e^{-i\mu t} & -e^{-i\mu t} & 0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\e^{i\mu t}\\-e^{i\mu t}\\0\end{pmatrix} = \frac{1}{3}\cos\mu t$$

$$\langle S_y \rangle = \langle \psi_\beta(t)|S_y^B|\psi_\beta(t)\rangle = \frac{i}{6}\begin{pmatrix}1 & e^{-i\mu t} & -e^{-i\mu t} & 0\end{pmatrix}\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ e^{i\mu t}\\ -e^{i\mu t}\\ 0\end{pmatrix} = \frac{1}{3}\sin\mu t$$

$$\langle S_z \rangle = \langle \psi_\beta(t)|S_z^B|\psi_\beta(t)\rangle = \frac{1}{6}\begin{pmatrix}1 & e^{-i\mu t} & -e^{-i\mu t} & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\ e^{i\mu t}\\ -e^{i\mu t}\\ 0\end{pmatrix} = \frac{1}{6}$$

$$\Rightarrow \overrightarrow{M_\beta}(t) = (\frac{1}{3}\cos\mu t, \frac{1}{3}\sin\mu t, \frac{1}{6})$$