E4854: Dynamics of coupled spins

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The problem:

Two coupled spins are described by the Hamiltonian:

$$\mathcal{H} = \mu S^A \cdot S^B + c S_x^A S_x^B + h(S_z^A + S_z^B)$$

where $\vec{S} = \frac{1}{2}\vec{\sigma}$.

The standard basis for representation is $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

(1) Write in the standard basis the three matrices that represent the three components of the Hamiltonian.
(2) The two spins are prepared in up mode. What is the rotational frequency of the system?
(3) One spin is prepared in up mode, and the second in down mode. What is the rotational frequency of the system?
(4) Schematically draw the dependence of the eigenvalues in the magnetic field h. Assume that $0 < \mu, c, h$.
(5) The system is prepared in the ground state, with zero magnetic field. We now slowly increase the magnetic field to a very large value. In the standard basis, write what is the initial state and the final state of the system.
(6) In practice, the system is not isolated from the environment and weak interaction can exist:

$$\mathcal{H}_{int} = (S_x^A, S_y^A, S_z^A, S_x^B, S_y^B, S_z^B)$$

Can this interaction affect the final state of the system? Specify what is the final state if there is an effect. If there is no effect, briefly explain why not.

The solution:

(1) We will denote: $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

From here, we can represent the standard basis: $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

$$|\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Using the standard basis we can represent $\overrightarrow{S^A}$ and $\overrightarrow{S^B}$:

$$S_x^A = S_x \otimes \mathbb{I}_{2\times 2} = \frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, S_x^B = \mathbb{I}_{2\times 2} \otimes S_x = \frac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$S_y^A = \frac{i}{2}\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}, S_y^B = \frac{i}{2}\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

$$S_z^A = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ S_z^B = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

The Hamiltonian is composed of 3 parts:
$\mathcal{H} = \mathcal{H}_\mu + \mathcal{H}_c + \mathcal{H}_h$
Each part can be represented in the standard basis using the matrices $\overrightarrow{S^A}$, $\overrightarrow{S^B}$:

$$\mathcal{H}_\mu = \mu\overrightarrow{S^A}\cdot\overrightarrow{S^B} = \mu(S_x^A\cdot S_x^B + S_y^A\cdot S_y^B + S_z^A\cdot S_z^B) = \frac{\mu}{4}\begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$$

$$\mathcal{H}_c = cS_x^AS_x^B = \frac{c}{4}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \mathcal{H}_h = h(S_z^A+S_z^B) = h\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix}$$

(2) The initial state of the system is $|\psi_{(t=0)}\rangle = |\uparrow\uparrow\rangle$.
From the previous section we can see that the Hamiltonian is block-diagonaled. There are only two pairs of coupled states: $|\uparrow\uparrow\rangle$ is coupled with $|\downarrow\downarrow\rangle$ and $|\uparrow\downarrow\rangle$ is coupled with $|\downarrow\uparrow\rangle$. Thus, we can write the reduced Hamiltonian $\mathcal{H}_1$of states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ as:

$$\mathcal{H}_1 = \frac{\mu}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{c}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix} + h\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\mu}{4} + \overrightarrow{\Omega}\cdot\overrightarrow{S}$$

where $\overrightarrow{\Omega} = 2(\frac{c}{4},0,h)$.
Therefore the rotational frequency of the system is:

$$\Omega = \sqrt{(\tfrac{c}{2})^2 + (2h)^2}$$

(3) In this case, the initial state of the system is $|\psi(t=0)\rangle = |\uparrow\downarrow\rangle$.
The reduced Hamiltonian $\mathcal{H}_2$ of states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ is:

$$\mathcal{H}_2 = \frac{\mu}{4}\begin{pmatrix}-1&2\\2&-1\end{pmatrix} + \frac{c}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix} + h\begin{pmatrix}0&0\\0&0\end{pmatrix} = -\frac{\mu}{4}\mathbb{I} + \frac{\mu}{2}\sigma_x + \frac{c}{4}\sigma_x = -\frac{\mu}{4}\mathbb{I} + \left(\frac{\mu}{2}+\frac{c}{4}\right)\sigma_x = -\frac{\mu}{4} + \overrightarrow{\Omega}\cdot\overrightarrow{S}$$

where $\overrightarrow{\Omega} = 2\left(\frac{\mu}{2}+\frac{c}{4},0,0\right)$.
Therefore the rotational frequency of the system is:
$\Omega = \mu + \frac{c}{2}$

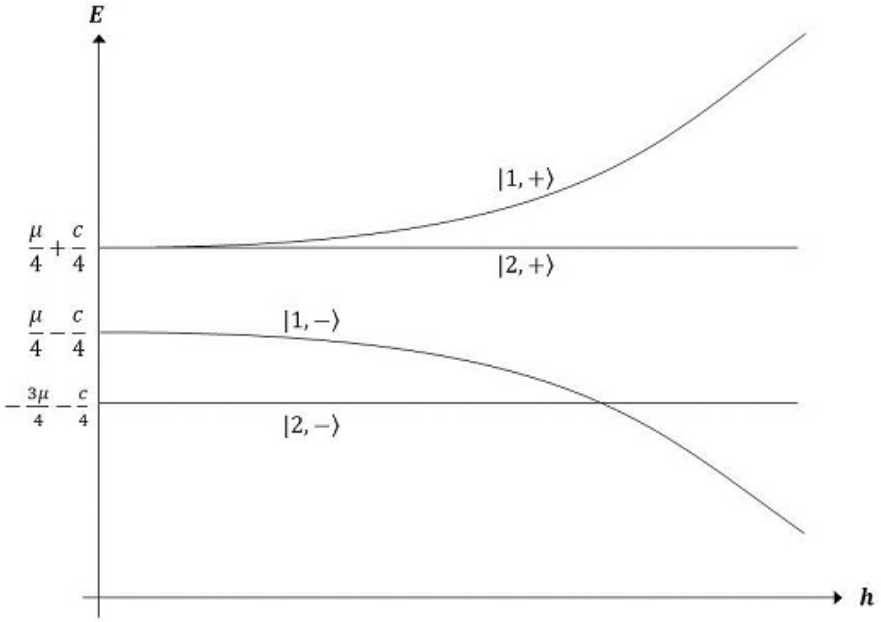
(4) The eigen energies of $\mathcal{H}_1$and $\mathcal{H}_2$ are:

$$E_1 = \frac{\mu}{4} \pm \sqrt{(\tfrac{c}{4})^2 + h^2},\ E_2 = -\frac{\mu}{4} \pm \frac{\mu}{2} + \frac{c}{4}$$

Or more explicetly:

$E_{1,+} = \frac{\mu}{4} + \sqrt{(\frac{c}{4})^2 + h^2}$
$E_{1,-} = \frac{\mu}{4} - \sqrt{(\frac{c}{4})^2 + h^2}$
$E_{2,+} = \frac{\mu}{4} + \frac{c}{4}$
$E_{2,-} = -\frac{3\mu}{4} - \frac{c}{4}$ (singlet)

Figure 1: Eigen energies vs. h



(5) For h=0, the ground state is $|2,-\rangle$ with energy $E_{2,-} = -\frac{3\mu}{4} - \frac{c}{4}$. This state is coupled only to the state $|2,+\rangle$, which his energy is h independent.
Therefore, the initial state in the standard basis is $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ (Singlet State), and it is also the final state.

(6) In case that H_{int} will couple between $|1,-\rangle$ and $|2,-\rangle$ or between $|1,+\rangle$ and $|2,-\rangle$, the final state of the system will change to $|1,-\rangle$. This will happen because $E_{1,-}$ is h dependent and decreases as the value of h increases. Since we start at the ground state, and we use an adiabatic approximation, we will remain at the ground state.
For large values of h, we get $h \gg c, \mu$, which suggests that $E_{1,\pm} \approx \pm h$. In this case, the Hamiltonian becomes $\mathcal{H} \approx \mathcal{H}_h$, with no coupling between $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. Therefore, the final state of the system is $|1,-\rangle \approx |\downarrow\downarrow\rangle$.