E4852: Dynamics of coupled spins

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The problem:

Two coupled spins are described by the Hamiltonian $H = \mu \vec{S}^A \cdot \vec{S}^B + const$, when $\vec{S} = \frac{1}{2}\vec{\sigma}$. We prepare the first spin in Z polarization state and the second spin in X polarization state. Without loss of generality we focus here on the movement of the second spin ($\vec{S}^B$). The spin state is described by the polarization vector $\vec{M} = (\langle\sigma_x^B\rangle, \langle\sigma_y^B\rangle, \langle\sigma_z^B\rangle)$. Tip: In the calculations required in sections 4-5, it is convenient to determine the Constant in the Hamiltonian, so that the energy in the Triplet states would be zero.

(1) If it was possible to freeze the first spin's state, what was the precession frequency of the second spin? And what was the result for the polarization vector $\vec{M}(t)$?

(2) Write the initial state of the system as a superposition of Singlet and Triplet states.

(3) Write down the Hamiltonian at the Singlet-Triplet base. What is the motion frequency Ω of the system?

(4) What is the result for $M_z(t)$?

(5) What is the result for $M_x(t), M_y(t)$?

(6) What is the result for the length of the polarization vector? Is it constant?

(7) In what times the spin can be described by a "wave function"?

The solution:

(1) It is given that the first spin's state remains unchanged, therefore, $\vec{S}^A$ must rotate the first spin along the Z axis so one can write $\vec{S}^A = (0, 0, S_z)$. After inserting $\vec{S}^A$ into the Hamiltonian and assuming $const = 0$ one gets:

$$H = \mu S^A \cdot S^B + const = \mu S_z^A S_z^B = \frac{1}{2}\mu \sigma_z^A S_z^B$$

And using the fact that $H = \vec{\Omega} \cdot \vec{S}$ one gets:

$$\vec{\Omega} = (0, 0, \frac{\mu}{2}); \quad |\vec{\Omega}| = \frac{\mu}{2}$$

The polarization vector is defined by $\vec{M} = (\langle\sigma_x^B\rangle, \langle\sigma_y^B\rangle, \langle\sigma_z^B\rangle)$. In order to calculate it one should first write the quantum state for $t = 0$:

$$|\psi^0\rangle = |\uparrow\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

And now one can proceed:

$$\sigma_x^B = \hat{I} \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \langle \sigma_x^B \rangle = \langle \Psi^0 | \sigma_x^B | \Psi^0 \rangle = 1$$

$$\sigma_y^B = \hat{I} \otimes \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \Rightarrow \langle \sigma_y^B \rangle = \langle \Psi^0 | \sigma_y^B | \Psi^0 \rangle = 0$$

$$\sigma_z^B = \hat{I} \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \langle \sigma_z^B \rangle = \langle \Psi^0 | \sigma_z^B | \Psi^0 \rangle = 0$$

So one obtains:

$$\vec{M}(0) = (1, 0, 0)$$

In order to calculate $\vec{M}(t)$ one should use $\vec{M}(t) = R^E(\Phi) \cdot \vec{M}(0)$ while $\Phi = \vec{\Omega} t$ and $\vec{\Omega} = \Omega \hat{z}$ so:

$$R^E(\vec{\Omega} t) = R_z^E(\Omega t) = \begin{pmatrix} cos(\Omega t) & -sin(\Omega t) & 0 \\ sin(\Omega t) & cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And finally:

$$\vec{M}(t) = R_z^E(\Omega t) \cdot \vec{M}(0) = (cos(\Omega t), sin(\Omega t), 0)$$

(2) The Singlet - Triplet states are:

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$|1,1\rangle = |\uparrow\uparrow\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle$$

So one can write the initial state of the system as:

$$|\psi^0 >= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{2}(|1,0\rangle + |0,0\rangle)$$

(3) Transferring the Hamiltonian to the Singlet-Triplet base:

$$H|0,0\rangle = \frac{\mu}{4}\vec{\sigma}^A \cdot \vec{\sigma}^B[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)] = -\frac{3}{4}\mu|0,0\rangle$$

$$H|1,1\rangle = \frac{\mu}{4}\vec{\sigma}^A \cdot \vec{\sigma}^B|\uparrow\uparrow\rangle = \frac{\mu}{4}|1,1\rangle$$

$$H|1,0\rangle = \frac{\mu}{4}\vec{\sigma}^A \cdot \vec{\sigma}^B[\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)] = \frac{\mu}{4}|1,0\rangle$$

$$H|1,-1\rangle = \frac{\mu}{4}\vec{\sigma}^A \cdot \vec{\sigma}^B|\downarrow\downarrow\rangle = \frac{\mu}{4}|1,-1\rangle$$

So eventually one can write the new Hamiltonian:

$$H_{new} = \begin{pmatrix} -\frac{3}{4}\mu & 0 & 0 & 0 \\ 0 & \frac{\mu}{4} & 0 & 0 \\ 0 & 0 & \frac{\mu}{4} & 0 \\ 0 & 0 & 0 & \frac{\mu}{4} \end{pmatrix}$$

By exploiting the freedom of the constant in the Hamiltonian one can easily obtain that $\Omega = \mu$.

(4) One can define the constant in the Hamiltonian as $const = -\frac{\mu}{4}$ and get:

$$H_{new} = \begin{pmatrix} -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The time dependant quantum state will be:

$$|\psi^t\rangle = \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{2}|1,0\rangle + \frac{1}{2}e^{i\mu t}|1,0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1/2 + 1/2e^{i\mu t} \\ 1/2 - 1/2e^{i\mu t} \\ 0 \end{pmatrix}$$

And, therefore, one can calculate $M_z(t)$ and get (after some algebra):

$$M_z(t) = \langle\psi^t|\sigma_z^B|\psi^t\rangle = \frac{1}{2}(1 - cos(\Omega t))$$

(5) In the same way one can calculate $M_x(t)$ and $M_y(t)$ and get:

$$M_x(t) = \langle\psi^t|\sigma_x^B|\psi^t\rangle = \frac{1}{2}(1 + cos(\Omega t))$$

and

$$M_y(t) = \langle\psi^t|\sigma_y^B|\psi^t\rangle = \frac{1}{2}sin(\Omega t)$$

(6) The polarization vector is $\vec{M(t)} = (M_x(t), M_y(t), M_z(t))$ so one can use sections 4-5 and write:

$$|\vec{M}(t)|^2 = M_x(t)^2 + M_y(t)^2 + M_z(t)^2 = 1 - \frac{1}{4}sin^2(\Omega t)$$

It is easy to see that the polarization vector's length is time-dependant so it is NOT a constant.

(7) By examining the polarization vector one can see that its length is not a multiple of 1/2 and, therefore, we have an entangled quantum state. For the ability to describe the spin as a wave function one must demand that $|\vec{M}(t)| = \frac{1}{2} \cdot integer$ and get the condition:

$$t = \frac{\pi}{\Omega} \cdot integer$$