Ex4832: Three Spin-1 Particle Interaction

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The problem:

The Hamiltonian governing the evolution of a three spin-1 particle system is given by:

$$\mathcal{H} = \epsilon\Bigg(\vec{S_1}\cdot\vec{S_2}+\vec{S_1}\cdot\vec{S_3}+\vec{S_2}\cdot\vec{S_3}\Bigg)+g\vec{S_1}\cdot\vec{S_3}+\vec{h}\big(\vec{S_1}+\vec{S_2}+\vec{S_3}\big)$$

Where the coefficients $\epsilon, g, h > 0$.

The total angular momentum is a constant of motion. As a result it is convenient to describe the eigenstates of the system in the basis of the total angular momentum,denoted as $|J, M, l\rangle$

The standard basis is described by the state $|m_1, m_2, m_3\rangle$, where $m_i = 0 \pm 1$ and i=1,2,3

(1) What is the decomposition of $3 \otimes 3 \otimes 3 =$?
(2) What are the eigenvalues λ_l of $\vec{S_1}\cdot\vec{S_3}$?
(3) Give an explicit expression of the energies of the system $E_{J,M,l} =$?
(4) What is the maximum energy of the system? $E_{Max} =$???
(5) Express the state$|J = 3, M = 2\rangle$ in the standard basis $|m_1, m_2, m_3\rangle$
(6) Write orthogonal states that possess exchange symmtery between particles $1 \longleftrightarrow 3$.

$$|J = 2, M = 2, \pm\rangle = ?$$

(7) Find the energies of these respective states? $E_{J=2,M=2,\pm} =$?, and $E_{J=3,M=2,} =$?

Guide: First notice the Hamiltonian can be described by the total angular momentum $\vec{J}$ and proceed from there. The idea is to think of good quantum numbers to work with.The solution should be based on deductive reasoning, furthermore it is not advisable to work with matrix calculations.

The solution:

(1) To develop the fundamental principles of how to decompose a representation, we will begin with the simple example $3 \otimes 3$. It is important bear in mind that we are changing basis. Initially we are transferring from the standard basis , $|m_1, m_3\rangle$, to the new basis, $|l, m_l\rangle$. The new basis corresponds to the irreducible matrix representations where the matrix representation is known. The dimension of the matrices uniquely determine the matrix representation. Because $dim = 2j + 1$ it follows j=1 for a spin-1 particle. As a result the product space of two spin-1 particles is represented by a $9\times$ 9 matrix.(To verify consistency the dimensions of the direct sum of the decomposed representations must add up 9.)

In general the external product between two matrices decomposes, according to the "Addition of Angular Momentum Theorem", in the form:

$$(2s_1 + 1) \otimes (2s_2 + 1) = (2|s_1 + s_2| + 1) \oplus \cdots \oplus (2|s_1 - s_2| + 1)$$
$$s_1 = 1, s_2 = 1 \rightarrow 3 \otimes 3 = 5 \oplus 3 \oplus 1$$

The total spin angular momentum of particles 1 and 3 is denoted by the quantum number "l". For two spin-1 particles the possible values "l" can obtain are $l = 2, 1, 0$. Therefore $3 \otimes 3$ matrix reduces into three irreducible representations with dimensions 5,3,1 accordingly.

Using the same the technique we can decompose $3 \otimes 3 \otimes 3$ into a direct sum of irreducible representations.

Remark: The general idea is as follows : We begin with the basis $|m_1, m_2, m_3\rangle$. Then we transfer to the basis of $|l, m_l, m_2\rangle$. Finally we transfer to the final basis $|J, l, M\rangle$

$$3 \otimes 3 \otimes 3 = 3 \otimes \big(5 \oplus 3 \oplus 1\big)$$
$$3 \otimes 3 \otimes 3 = 7 \oplus 5 \oplus 3 \oplus 5 \oplus 3 \oplus 1 \oplus 3$$

(2) As previously mentioned, we will transfer to a basis,$|l\rangle$, which contains the total spin angular momentum $\hat{L} = \vec{S_1} + \vec{S_3}$ of particles 1 and 3. We will express $\vec{S_1} \cdot \vec{S_3}$ in terms of good quantum numbers $|l, m_l\rangle$

$$\hat{L^2} = \big(\vec{S_1} + \vec{S_3}\big)^2 = \vec{S_1}^2 + \vec{S_3}^2 + 2\vec{S_1} \cdot \vec{S_3} \tag{1}$$

Remark: The matrices $\vec{S_1}$ and $\vec{S_3}$ commute

$$\frac{1}{2}\left(\hat{L^2} - \vec{S_1}^2 - \vec{S_3}^2\right) = \vec{S_1} \cdot \vec{S_3} \tag{2}$$

$$\hat{L^2}|l\rangle = l(l+1)|l\rangle$$
$$\vec{S_i}^2|l\rangle = s_i(s_i+1)|l\rangle = 2|l\rangle \qquad since \quad s_i = 1$$

Therefore

$$\vec{S_1} \cdot \vec{S_3}|l\rangle = \frac{1}{2}\left(\hat{L^2} - \vec{S_1}^2 - \vec{S_3}^2\right)|l\rangle = \frac{1}{2}\Big(l(l+1) - 4\Big)|l\rangle \tag{3}$$

Recall "l" is uniquely determined by the dimension of the irreducible representation. Because "l" obtains the values 2,1,0, it follows that the eigenvalues of $\vec{S_1} \cdot \vec{S_3}$ are:

$$\lambda_0 = -2 \tag{4}$$
$$\lambda_1 = -1 \tag{5}$$
$$\lambda_2 = 1 \tag{6}$$

(3) First we will orient our axes in such a way that that the Magnetic field $\vec{h}$ coincides with the z-axis. This is justified due to spherical symmetry. The motivation for rotating our axis in such a way is mainly to simplify the mathematics and to work in the basis of good quantum numbers $|J, M, l\rangle$. The surviving terms in dot product between the magnetic field and individual spins are only the z-component of the individual spin angular momentum.

The total spin angular momentum is defined as:

$$\vec{J} = \vec{S_1} + \vec{S_2} + \vec{S_3}$$

Note:

$$\hat{J}^2 = \left(\vec{S_1} + \vec{S_2} + \vec{S_3}\right)^2 = S_1^2 + S_2^2 + S_3^2 + 2\left(\vec{S_1}\cdot\vec{S_2} + \vec{S_1}\cdot\vec{S_3} + \vec{S_2}\cdot\vec{S_3}\right) \tag{7}$$

$$\vec{S_1}\cdot\vec{S_2} + \vec{S_1}\cdot\vec{S_3} + \vec{S_2}\cdot\vec{S_3} = \frac{1}{2}\left(\hat{J}^2 - \left(S_1^2 + S_2^2 + S_3^2\right)\right) \tag{8}$$

The Hamiltonian expressed in the basis $|J, M, l\rangle$ is given by:

$$\mathcal{H} = \frac{\epsilon}{2}\left(\hat{J}^2 - 6\right) + \frac{g}{2}\left(\hat{L}^2 - 4\right) + h\hat{J}_z \tag{9}$$

$$\begin{aligned}
&\hat{J}^2|J, M, l\rangle + J(J+1)|J, M, l\rangle \\
&\hat{J}_z|J, M, l\rangle = (m_1 + m_2 + m_3)|J, M, l\rangle; \qquad M = (m_1 + m_2 + m_3) \\
&\hat{L}^2|J, M, l\rangle = l(l+1)|J, M, l\rangle
\end{aligned}$$

The energies of the system $E_{J,M,l}$ are:

$$E_{J,M,l} = \frac{\epsilon}{2}\left(J(J+1) - 6\right) + \frac{g}{2}\big(l(l+1) - 4\big) + hM \tag{10}$$

(4) The largest irreducible matrix corresponds to a dim=7 representation, which corresponds to J=3. Consequently it is obvious that the penthouse state is $|J = 3, M = 3, l = 2\rangle$. As a result, the maximum energy is given by:

$$E_{3,3,2} = 3\epsilon + g + 3h \tag{11}$$

(5) We want to express the state $|J = 3, M = 2\rangle$ in terms of the standard basis $|m_1, m_2, m_3\rangle$. The idea is to use the ladder operator on the penthouse state in order to achieve the multiplet of states $|J = 3, M = 2\rangle$

$$\hat{S_-} = \hat{S_-^1} + \hat{S_-^2} + \hat{S_-^3} \tag{12}$$

$$|J = 3, M = 3, l = 2\rangle = |m_1 = 1, m_2 = 1, m_3 = 1\rangle \tag{13}$$

$$\hat{S_-}|J = 3, M = 3\rangle = \left(\hat{S_-^1} + \hat{S_-^2} + \hat{S_-^3}\right)|m_1 = 1, m_2 = 1, m_3 = 1\rangle \tag{14}$$

Thus the state $|J = 3, M = 2\rangle$ is given by:

$$|J = 3, M = 2\rangle = \frac{1}{\sqrt{3}}\left(|1, 1, 0\rangle + |0, 1, 1\rangle + |1, 0, 1\rangle\right) \tag{15}$$

(6) We are looking for orthogonal states $|J = 2, M = 2, \pm\rangle$ which possess exchange symmetry between particles $1 \longleftrightarrow 3$. By inspection the states with the exchange symmetry of particles $1 \longleftrightarrow 3$ are:

$$|0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{16}$$

The anti-symmetric state is:

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{17}$$

We will build an orthogonal state $|+\rangle$ that is symmetric to particle exchange. In order to do so we require the inner product with the states $|-\rangle$, and $|0\rangle$ to vanish. We define the state $|+\rangle$ as an orthogonal state symmetric to particle exchange:

$$|+\rangle = \begin{pmatrix} \zeta \\ 1 \\ \zeta \end{pmatrix} \tag{18}$$

Obviously $\langle -|+\rangle = 0$, thus to find ζ we will examine the inner product $\langle +|0\rangle = 0$

$$2\zeta + 1 = 0 \to \zeta = -\frac{1}{2} \tag{19}$$

After normalization $|+\rangle$

$$|+\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{20}$$

(7)
According to equation (9)the energies of the states are:
$|J = 3, M = 2\rangle$ has energy

$$E_{3,2,2} = 3\epsilon + g + 2h \tag{21}$$

$|J = 2, M = 2, -\rangle$ has energy

$$E_{2,2,-} = -g + 2h \tag{22}$$

$|J = 2, M = 2, +\rangle$ has energy

$$E_{2,2,+} = +g + 2h \tag{23}$$