E486: Addition angular momentum of spin one and spin half

Submitted by: Tal Saggi

The problem:

(1) Reduce the standard basis of spin $S_1 = 1$ plus spin $S_2 = \frac{1}{2}$
(2) Write the transformation matrix between the old basis and the new basis
(3) Find the matrix elements of the J^2 operator in the old basis

The solution:

(1) The standard basis of spin $S_1 = 1$ is

$$|S_1, S_{1z}\rangle : \quad \{|1,1\rangle \quad |1,0\rangle \quad |1,-1\rangle\}$$

The standard basis of spin $S_2 = \frac{1}{2}$ is

$$|S_2, S_{2z}\rangle : \quad \{|\frac{1}{2},\frac{1}{2}\rangle \quad |\frac{1}{2},-\frac{1}{2}\rangle\}$$

Addition the two spins will generate a dim $= 3 \otimes 2 = 6$ dimensional space.

$$|S_1, S_2, S_{1z}S_{2z}\rangle \longmapsto |S_{1z}, S_{2z}\rangle : \quad \{|1,\frac{1}{2}\rangle \quad |1,-\frac{1}{2}\rangle \quad |0,\frac{1}{2}\rangle \quad |0,-\frac{1}{2}\rangle \quad |-1,\frac{1}{2}\rangle \quad |-1,-\frac{1}{2}\rangle\}$$

Because of $\quad m_j = S_{1z} + S_{2z} \quad$ the values of m_j are $\quad \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

In additional $\quad -j \le m_j \le j \quad$ in jumps of one and $\quad |S_1 - S_2| \le j \le |S_1 + S_2| \quad$ makes the new basis:

$$|j, m_j\rangle : \quad \{|\frac{3}{2},\frac{3}{2}\rangle \quad |\frac{3}{2},\frac{1}{2}\rangle \quad |\frac{3}{2},-\frac{1}{2}\rangle \quad |\frac{3}{2},-\frac{3}{2}\rangle \quad |\frac{1}{2},\frac{1}{2}\rangle \quad |\frac{1}{2},-\frac{1}{2}\rangle\}$$

The subspace $j = \frac{1}{2}$ have dim $= 2$

The subspace $j = \frac{3}{2}$ have dim $= 4$

Therefore the new basis keep dim $= 2 \oplus 4 = 6$.

(2) By applying J_- operator on the old basis and keeping the normalization we generat the new basis.

$$J_-|j, m_j\rangle = \sqrt{j\cdot(j+1) - m_j\cdot(m_j-1)}|j, m_j-1\rangle \qquad J_- = L_- + S_-$$

$$|\frac{3}{2},\frac{3}{2}\rangle = |1,\frac{1}{2}\rangle$$

$$J_-|\frac{3}{2},\frac{3}{2}\rangle = J_-|1,\frac{1}{2}\rangle = L_-|1,\frac{1}{2}\rangle + S_-|1,\frac{1}{2}\rangle = \sqrt{2}|0,\frac{1}{2}\rangle + |1,-\frac{1}{2}\rangle$$

$$\Rightarrow |\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1,-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|0,\frac{1}{2}\rangle$$

$$J_-|\frac{3}{2},\frac{1}{2}\rangle = L_-\sqrt{2}|0,\frac{1}{2}\rangle + S_-\sqrt{2}|0,\frac{1}{2}\rangle + L_-|1,-\frac{1}{2}\rangle + S_-|1,-\frac{1}{2}\rangle = 2\cdot\sqrt{2}|0,-\frac{1}{2}\rangle + 2|-1,\frac{1}{2}\rangle$$

$$\Rightarrow |\frac{3}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|-1,\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|0,-\frac{1}{2}\rangle$$

$$J_-|\frac{3}{2},-\frac{1}{2}\rangle = S_- \quad 2|-1,\frac{1}{2}\rangle + L_- \quad 2\cdot\sqrt{2}|0,-\frac{1}{2}\rangle = 6|-1,-\frac{1}{2}\rangle$$

$$\Rightarrow |\frac{3}{2},-\frac{3}{2}\rangle = |-1,-\frac{1}{2}\rangle$$

By writeing the $j=\frac{1}{2}$ subspace vectors as linear combination of old basis vectors with $S_{1z}+S_{2z}=\pm\frac{1}{2}$ and keeping the orthogonalization we get:

$$\begin{aligned}
|\frac{1}{2},\frac{1}{2}\rangle &= -\sqrt{\frac{2}{3}}|1,-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|0,\frac{1}{2}\rangle \\
|\frac{1}{2},-\frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}}|0,-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|-1,\frac{1}{2}\rangle
\end{aligned}$$

The transformation matrix between the old basis and the new basis $T_{lk}=\langle S_{1z},S_{2z}|j,m_j\rangle$ is:

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\
0 & \sqrt{\frac{2}{3}} & 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & -\sqrt{\frac{1}{3}} \\
0 & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

(3) In the new basis $J^2|j,m_j\rangle = j\cdot(j+1)|j,m_j\rangle$
The matrix elements of the J^2 operator in the new basis is:

$$\langle j_l, m_{j_l}|J^2|j_k, m_{j_k}\rangle = \delta_{lk}\cdot j_k\cdot(j_k+1)$$

$$J^2 = \begin{pmatrix}
\frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{15}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{15}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{15}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{4}
\end{pmatrix}$$

In the old basis the matrix elements of the J^2 operator will be:

$$J^2{}_{old} = T\cdot J^2{}_{new}\cdot T^\dagger = \begin{pmatrix}
\frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{7}{4} & \sqrt{2} & 0 & 0 & 0 \\
0 & \sqrt{2} & \frac{11}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{11}{4} & \sqrt{2} & 0 \\
0 & 0 & 0 & \sqrt{2} & \frac{7}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{15}{4}
\end{pmatrix}$$