E483: Singlet and triplet states

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The problem:

Show that the singlet and triplet states are eigenstates of $\hat{J}^2$ and $\hat{J}_z$.

The solution:

(1) First we should find the matrix representation of $\hat{J}^2$ and $\hat{J}_z$ in the standard base.
The standard base is:

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(2) We find the $\hat{J}_z$ matrix by external multiplication:

$$\hat{J}_z = \hat{1} \otimes \sigma_3 + \sigma_3 \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The $\hat{J}^2$ matrix could be found by matrix multiplication (see lecture notes 17.7) using the transformation matrix T:

$$\hat{J}^2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(3) Writing the triplet and singlet states in the standard base gives:
A. Triplet:

$$|1,1\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

B. Singlet:

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

(4) Finally we show that the singlet and triplet states are eigenstates of $\hat{J}^2$ and $\hat{J}_z$:
A. For $\hat{J}^2$:

$$\hat{J}^2|1,1\rangle = 2|1,1\rangle$$

$$\hat{J}^2|1,0\rangle = 2|1,0\rangle$$

$$\hat{J}^2|1,-1\rangle = 2|1,-1\rangle$$

$$\hat{J}^2|0,0\rangle = 0|0,0\rangle$$

B. For $\hat{J}_z$:

$$\hat{J}_z|1,1\rangle = 1|1,1\rangle$$

$$\hat{J}_z|1,0\rangle = 0|1,0\rangle$$

$$\hat{J}_z|1,-1\rangle = -1|1,-1\rangle$$

$$\hat{J}_z|0,0\rangle = 0|0,0\rangle$$