E4660: Building Y^{lm}

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The problem:

For angular momentum we define: $\vec{L} = \vec{r} \times \vec{p}$. Components of this angular momentum satisfy the following: $[L_i, L_j] = i\epsilon_{ijk}L_k$.
From these comutation relations it follows that L^2 has eigenvalues which are either integers or half integers.
On the other hand we know that in spatial representation we have :

$$L_z = -i\frac{\partial}{\partial\phi} \tag{1}$$

$$L_x = -i\left(-sin(\theta)\frac{\partial}{\partial\theta} - cot(\theta)cos(\phi)\frac{\partial}{\partial\phi}\right) \tag{2}$$

$$L_y = -i\left(cos(\theta)\frac{\partial}{\partial\theta} - cot(\theta)sin(\phi)\frac{\partial}{\partial\phi}\right) \tag{3}$$

(1) Find Y^{11} from the requirement: $L_+|1,1\rangle = 0$.
(2) Find the rest of the Y^{lm} by using L_- .

The solution:

$$Y^{lm} = \langle\theta, \phi|lm\rangle$$

$$L_+ = L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + icot(\theta)\frac{\partial}{\partial\phi}\right)$$

Using:

$$L_+|1,1\rangle = 0 \qquad , \qquad L_z = -i\frac{\partial}{\partial\phi}$$

We get:

$$e^{i\phi}\left(\frac{\partial}{\partial\theta} + icot(\theta)\frac{\partial}{\partial\phi}\right)Y^{11} = e^{i\phi}\left(\frac{\partial}{\partial\theta} - cot(\theta)L_z\right)Y^{11}$$

Now using:

$$L_z|lm\rangle = m|lm\rangle$$

We get:

$$e^{i\phi}\left(\frac{\partial}{\partial\theta} - cot(\theta)\right)Y^{11} = 0$$

The solution to this equation is:

$$Y^{11} = Asin(\theta)f(\phi)$$

Using again $\quad L_z = -i\frac{\partial}{\partial\phi} \quad$ we get :

$$L_zY^{11} = -iAsin(\theta)\frac{\partial f(\phi)}{\partial\phi} = Asin(\theta)f(\phi) \Rightarrow Y^{11} = Asin(\theta)e^{i\phi}$$

Now we determine the normalization constant A :

$$\int |Y^{11}|^2 d\Omega = 1$$

And finally:

$$Y^{11} = -\sqrt{\frac{3}{8\pi}}sin(\theta)e^{i\phi}$$

(2) The rest can be obtained using :

$$L_- = (L_+)^\dagger$$

It follows that :

$$L_- = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + icot(\theta)\frac{\partial}{\partial\phi}\right)$$

And :

$$L_-|l,m\rangle = \sqrt{(l(l+1) - m(m-1)}|l,m-1\rangle$$

$$L_-|1,1\rangle = \sqrt{2}|1,0\rangle$$

In spatial representation :

$$Y^{10} = -\frac{1}{\sqrt{2}}e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + icot(\theta)\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{8\pi}}sin(\theta)e^{i\phi} = \sqrt{\frac{3}{4\pi}}cos(\theta)$$

The next would be obtained from the last as followed :

$$L_-|1,0\rangle = \sqrt{2}|1,-1\rangle$$

In spatial representaion :

$$Y^{1-1} = \frac{1}{\sqrt{2}}e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + icot(\theta)\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{4\pi}}cos(\theta) = \sqrt{\frac{3}{8\pi}}sin(\theta)e^{-i\phi}$$

Finally we get :

$$Y^{1-1} = \sqrt{\frac{3}{8\pi}}sin(\theta)e^{-i\phi}$$