E448: Dynamics in a two site system

Submitted by: Daniel Grossman

The problem:

A spinless particle is found in a two site system. Assume the amplitude of transition per unit time between the state $|\uparrow\rangle$ and the state $|\downarrow\rangle$ is Δ. In addition, there is an electrical field, so that the potential difference between the states is ϵ. At time $t = 0$ it is given that $P(t) = 1$.

(1) Write the Hamiltonian in the form $\hat{H} = \vec{\Omega} \cdot \hat{\vec{S}}$ (where $S_i = \frac{1}{2}\sigma_i$)
(2) Find $\vec{\Omega}$
(3) Express the precession frequency using ϵ,Δ
(4) Find the probability $P(t)$ of finding the particle at the state $|\uparrow\rangle$ at time t (use the z component of the magnetization)

The solution:

(1) Choosing the correct gauge Δ is real. The Hamiltonian in the position basis is:

$$\hat{\mathcal{H}} \mapsto \begin{pmatrix} \frac{\epsilon}{2} & \Delta \\ \Delta & -\frac{\epsilon}{2} \end{pmatrix} = \Delta\sigma_x + \frac{\epsilon}{2}\sigma_z = (2\Delta, 0, \epsilon) \cdot \hat{\vec{S}}$$

(2)

$$\vec{\Omega} = (2\Delta, 0, \epsilon)$$

(3) The time evolution operator assumes the form of a rotation:

$$\hat{U}(t) = \mathrm{e}^{-i\hat{\mathcal{H}}t} = \mathrm{e}^{-i\vec{\Omega}\cdot\hat{\vec{S}}t}$$

Therefore the precession frequency is:

$$\Omega = |\vec{\Omega}| = \sqrt{(2\Delta)^2 + \epsilon^2}$$

(4) The probability to find the particle at state $|\uparrow\rangle$ after time t given the initial conditions of the problem is found using the rotation formalism. In the general case a rotation is written as $\mathrm{e}^{-i\vec{\Phi}\cdot\hat{\vec{S}}} = \mathrm{e}^{-i\Phi\vec{n}\cdot\hat{\vec{S}}}$, where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector pointing in the direction of the rotation axis.
In our case putting $\Phi = \Omega t$, $\phi = 0$ and $\cos\theta = \frac{\epsilon}{\Omega}$ into the general formula for the rotation matrix $R(\Phi)$ (pp. 55-56 in the lecture notes - where we need only the component R_{11}) yields a final result of:

$$P(t) = |\langle\uparrow|\mathrm{e}^{-i\vec{\Omega}\cdot\hat{\vec{S}}t}|\uparrow\rangle|^2 = \left|\cos\left(\frac{\Phi}{2}\right) - i\cos\theta\sin\left(\frac{\Phi}{2}\right)\right|^2 = 1 - \frac{4\Delta^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)$$

We can see that if $\frac{\Delta}{\Omega} \ll 1$ then the probability for the particle to be found at $|\downarrow\rangle$ is very small for all t, which is just what one would expect.