E446: Representation of spin $\frac{1}{2}$ states

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The problem:

The quantum state of spin $\frac{1}{2}$ can be determined by 3 independent measurments. Assume these measurements are $M_j = \langle\sigma_j\rangle$, for j=1,2,3.

1. Express the probability matrix elements using the polarization vector $\vec{M}$ componets. Show that the resolt can be written as $\rho = \frac{1}{2}\left(1 + \vec{M}\cdot\vec{\sigma}\right)$.

2. Find the polarization vector $\vec{M}$ componets, for given probability matrices:

$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\rho = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

Introduction:

The Pauli matrices, together with the unit matrix, make a complete set in the 2-site (as in spin $\frac{1}{2}$) system. Hence, Any operator can be written as a linear combination of these matrices:

$\hat{A} = \mathrm{c}\hat{1} + \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z$

If we have measured $\sigma_{x,y,z}$, we can predict any other measurment by:

$\langle A\rangle = \mathrm{c} + \alpha\langle\sigma_x\rangle + \beta\langle\sigma_y\rangle + \gamma\langle\sigma_z\rangle$

One way of "packaging" the three independent measurments is the polarization vector:

$\vec{M} = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$.

The solution:

1. The probability matrix elements are:

$\mathrm{P}^{11} = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1 + \sigma_z)$

$\mathrm{P}^{12} = |\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y)$

$\mathrm{P}^{21} = |\downarrow\rangle\langle\uparrow| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y)$

$\mathrm{P}^{22} = |\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(1 - \sigma_z)$

Representing the probability matrix using $\vec{M}$ componets we get:

$\rho = \langle P^{ji}\rangle = \frac{1}{2}\begin{pmatrix} (1 + M_3) & (M_1 - iM_2) \\ (M_1 + iM_2) & (1 - M_3) \end{pmatrix} = \frac{1}{2}\left(1 + \vec{M}\cdot\vec{\sigma}\right)$.

2. Given ρ we can calculate $\vec{M}$ using the expectation value formula:

$\mathrm{M}_x = \langle \sigma_x \rangle = \sum\limits_{i,j} \sigma_{xi,j} \rho_{j,i} = trace(\sigma_x \rho)$.

$\mathrm{M}_y = trace(\sigma_y \rho)$.

$\mathrm{M}_z = trace(\sigma_z \rho)$.

Then we get:

$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longmapsto \vec{M} = (0, 0, 1)$.

$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \longmapsto \vec{M} = (1, 0, 0)$.

$\rho = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \longmapsto \vec{M} = (\frac{1}{2}, 0, -\frac{1}{2})$.

Now we can verify that indeed $\rho = \frac{1}{2}\left(1 + \vec{M} \cdot \vec{\sigma}\right)$, as proved.